Math 2280-1

Wednesday Oct 8

§3.6 cont'd.: forced oscillations

$$mx'' + cx' + kx = F_0 \cos\omega t$$

We did:

$$\left.\begin{array}{ll} \text{Case 1} & \omega \neq \omega_0,\ c=0 \\ & |\omega-\omega_0| \text{ large} \\ & |\omega-\omega_0| \text{ small (beating)} \\ \text{Case 2} & \omega=\omega_0 \text{ (resonance)} \\ & c=0 \end{array}\right\} \text{ finish examples page 4 Tuesday}$$

today: Case 3: $c \neq 0$ $(c>0)$
if $c \approx 0$, $\omega \approx \omega_0$ you may get "practical resonance"

also today: figure out ω_0 for more complicated mechanical systems

example 6 p.220: Solve

$$\begin{cases} x'' + 2x' + 26x = 82\cos 4t \\ x(0) = 6 \\ x'(0) = 0 \end{cases}$$

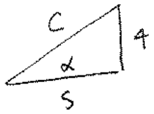
$x_H(t)$:

$x_p(t)$:

ans: $$x(t) = \underbrace{5\cos 4t + 4\sin 4t}_{C\cos(4t-\alpha);\ \boxed{x_{sp}(t)}} + \underbrace{e^{-t}(\cos 5t - 3\sin 5t)}_{\boxed{x_{tr}(t)}}$$

$C = \sqrt{41}$

$\alpha = \arctan(.8)$

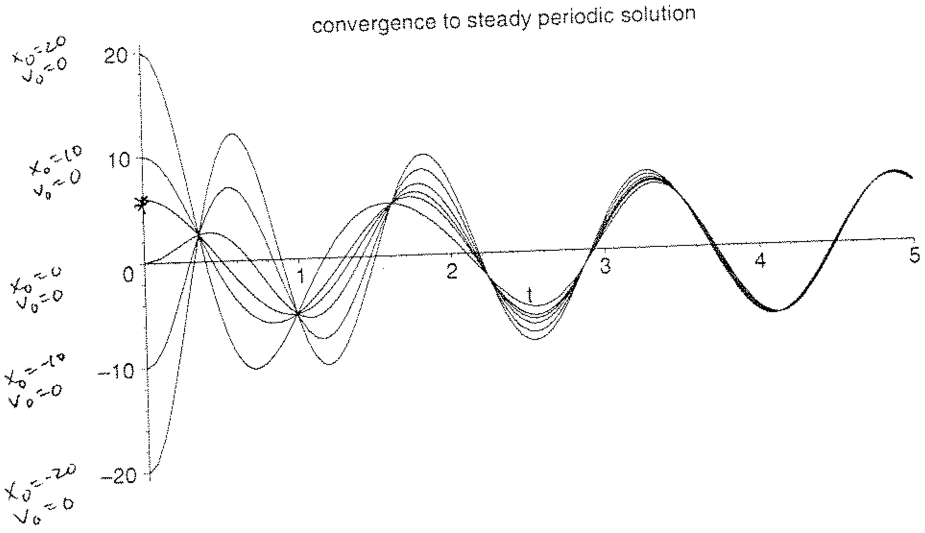


(the transient part of the soltn; decays as $t \to \infty$)

$x_{tr}(t)$ is determined by the initial conditions

the steady periodic part of the solution; is independent of initial conditions

see figure 3.6.8 p. 22[illegible], or below



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general case:

$$mx'' + cx' + kx = F_0\cos\omega t \qquad c \neq 0$$

$$\begin{array}{rl} k(& x_p = A\cos\omega t + B\sin\omega t \\ +\ c(& x_p' = -A\omega\sin\omega t + B\omega\cos\omega t \\ +\ m(& x_p'' = -A\omega^2\cos\omega t - B\omega^2\sin\omega t \end{array}$$

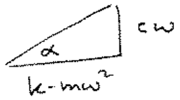
$$\begin{aligned} L(x_p) &= \cos\omega t\left[A(k-m\omega^2) + Bc\omega\right] & &= \cos\omega t\,[F_0] \\ &+ \sin\omega t\left[A(-c\omega) + B(k-m\omega^2)\right] & &+ \sin\omega t\,[0] \end{aligned}$$

$$\begin{bmatrix} k-m\omega^2 & c\omega \\ -c\omega & k-m\omega^2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{(k-m\omega^2)^2 + c^2\omega^2}\begin{bmatrix} k-m\omega^2 & -c\omega \\ c\omega & k-m\omega^2 \end{bmatrix}\begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{F_0}{(k-m\omega^2)^2 + c^2\omega^2}\begin{bmatrix} k-m\omega^2 \\ c\omega \end{bmatrix}$$

so $x_p(t) = C\cos(\omega t - \alpha)$



$$C = \sqrt{A^2+B^2}$$

$$C = \frac{F_0}{(k-m\omega^2)^2 + c^2\omega^2}\sqrt{(k-m\omega^2)^2 + c^2\omega^2}$$

$$\boxed{C = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + c^2\omega^2}}, \quad \alpha = \arccos\left(\frac{k-m\omega^2}{c}\right)}$$

notice denom can be very small if
$\omega \approx \omega_0$ (makes 1st term small)
$c \approx 0$ (makes second term small).

then

$$x(t) = \underbrace{C\cos(\omega t - \alpha)}_{x_{sp}(t)} + \underbrace{x_H(t)}_{x_{tr}(t)} = \begin{cases} c_1e^{-r_1t} + c_2e^{-r_2t} & \text{overdamped} \\ e^{-rt}(c_1 + c_2t) & \text{crit damped} \\ e^{-at}(A\cos bt + B\sin bt) & \text{underdamped.} \end{cases}$$

you are...

practical resonance occurs when C is $\gg F_0$,

i.e. when $\omega \approx \omega_0$, $c \approx 0$.

example:

$$x'' + 2x' + 26x = F_0 \cos \omega t$$

plot the fcn

$$\frac{1}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}, \text{ as a fcn of } \omega \quad \left(\begin{matrix} m = 1 \\ k = 26 \\ c = 2 \end{matrix} \right.$$

$$= \frac{1}{\sqrt{(26 - \omega^2)^2 + 4\omega^2}}$$

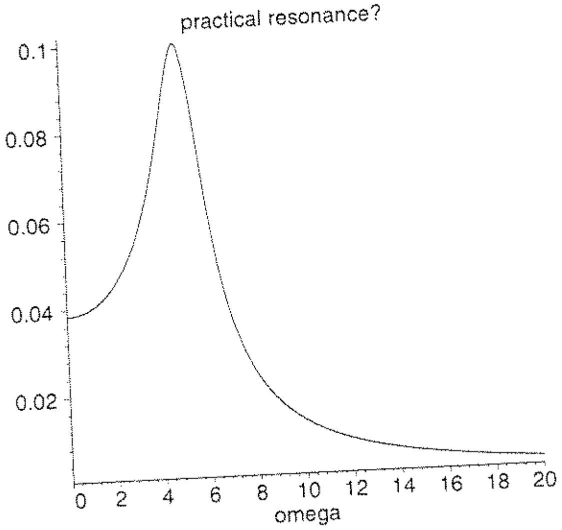
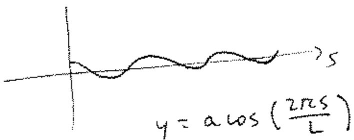


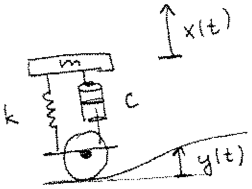
figure 3.6.9 p.221

unexpected
forced oscillation problems: p 218



$y = a\cos\left(\frac{2\pi s}{L}\right)$

road surface
period = L.



"car"

vertical acceleration:

$$mx'' = -k(x-y) \mp c(x'-y')$$

if $c=0$

$$mx'' + kx = ky$$

if speed of car is v then $s = vt$

$$= ka\cos\left(\frac{2\pi v}{L}t\right)$$

is a forced oscillation problem.
trouble when

$$\omega = \frac{2\pi v}{L} = \omega_0 = \sqrt{\frac{k}{m}}$$

$$v = \sqrt{\frac{k}{m}}\frac{L}{2\pi}$$

"washboard"

See also the front loading washing machine, #20 p. 222

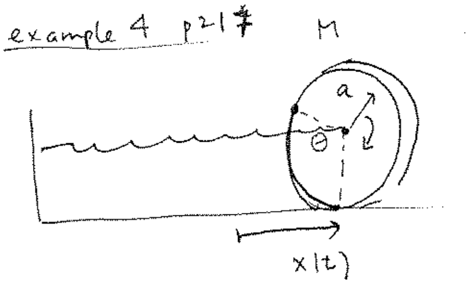
Using total energy to deduce natural angular frequency:
(for conservative systems) KE + PE = const

1. $KE = \frac{1}{2}mv^2$ mass m, speed v (of center of mass)
2. $KE = \frac{1}{2}I\omega^2$ rotating object (ω = angular vel.) I = moment of intertia.

3. $PE = \frac{1}{2}kx^2$ stretched spring
4 $PE = mgh$ gravitational PE

total energy is additive

example 4 p21~~8~~7



$$E = \frac{1}{2}kx^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}Mx'(t)^2$$

$x = a\theta$
$x' = a\theta' = a\omega$

I for solid wheel $= \frac{1}{2}Ma^2$ (you can work this out!)

so

$$E = \frac{1}{2}kx^2 + \frac{1}{4}Ma^2\left(\frac{x'}{a}\right)^2 + \frac{1}{2}M(x')^2$$

$$= \frac{1}{2}kx^2 + \frac{3}{4}M(x')^2$$

$$0 = \frac{dE}{dt} = kxx' + \frac{3}{2}Mx'x'' = x'\left[kx + \frac{3}{2}Mx''\right]$$

$$\boxed{\frac{3}{2}Mx'' + kx = 0}$$

$$\omega_0 = \sqrt{\frac{k}{\frac{3}{2}M}} = \sqrt{\frac{2k}{3M}}$$

moment of inertia for rotating disk

$$KE = \iint \frac{1}{2} \overset{v^2}{\overset{\downarrow}{(r\omega)^2}} \overbrace{\rho\, r\,dr\,d\theta}^{dm} \qquad \rho = \frac{M}{\pi a^2}$$

$$= \frac{1}{2}\rho\omega^2 \underbrace{\int_0^{2\pi} \underbrace{\int_0^a r^3\,dr}_{\frac{a^4}{4}}\,d\theta}_{\frac{\pi a^4}{2}}$$

$$= \cancel{\frac{1}{4}\pi\frac{M}{\pi a^2}}$$

$$= (\rho\pi a^2)\,\frac{1}{4}(a\omega)^2$$

$$= \frac{1}{4} M a^2\ \omega^2$$

$$= \frac{1}{2} I\omega^2 \quad \text{for } I = \frac{1}{2}Ma^2.$$