Math 2280-1

Tuesday Oct 7

§3.6 forced oscillations

most usual periodic forcing function

$$m x'' + c x' + k x = F_0 \cos\omega t$$

Case 1 $\quad c=0, \ \omega \neq \omega_0 \quad$ ($\omega_0 = \sqrt{\frac{k}{m}}$ still)

$$m x'' + k x = F_0 \cos\omega t$$

since $c=0$!

$+k[$ try $\quad x_p = A\cos\omega t + \cancel{B\sin\omega t}$

$+0[\quad x_p' = -A\omega\sin\omega t$

$+m[\quad x_p'' = -A\omega^2\cos\omega t$

$$L(x_p) = \cos\omega t\,[A][k - m\omega^2] \overset{\text{want}}{=} F_0\cos\omega t$$

$$A = \frac{F_0}{k - m\omega^2} \quad ; \quad \boxed{x_p(t) = \frac{F_0}{k - m\omega^2}\cos\omega t}$$

Now solve

$$\begin{cases} m x'' + c x' + k x = F_0 \cos\omega t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

$$x(t) = x_p + x_H = \frac{F_0}{k - m\omega^2}\cos\omega t + A\cos\omega_0 t + B\sin\omega_0 t$$

$$x(0) = x_0 = \frac{F_0}{k - m\omega^2} + A \quad \Rightarrow \quad A = x_0 - \frac{F_0}{k - m\omega^2}$$

$$x'(0) = v_0 = B\omega_0 \quad \Rightarrow \quad B = \frac{v_0}{\omega_0}$$

hence

$$\boxed{\begin{aligned} x(t) &= \frac{F_0}{k - m\omega^2}\left[\cos\omega t - \cos\omega_0 t\right] + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t \\ \text{or} \quad x(t) &= \frac{F_0}{m}\,\frac{1}{\omega_0^2 - \omega^2}\left[\cos\omega t - \cos\omega_0 t\right] + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t \end{aligned}}$$

Case 1a (boring) $|\omega-\omega_0|$ not small

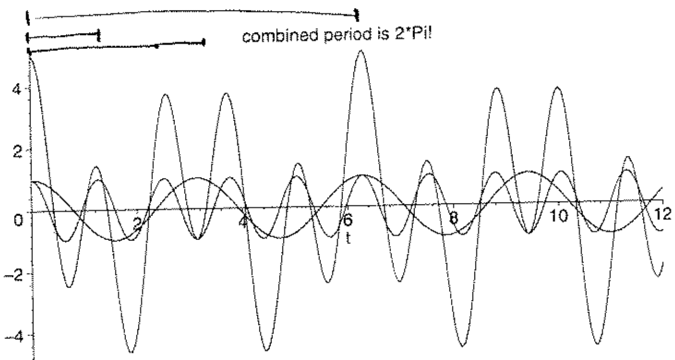
then from page 1,

$$x(t) = C_1 \cos\omega t + C_2\cos(\omega_0 t - \alpha)$$

period $T = \frac{2\pi}{\omega}$ period $T_0 = \frac{2\pi}{\omega_0}$

this superposition may or may not be periodic, depending on whether T and T_0 have a common (integer) multiple (iff T/T_0 is rational!)

```
> with(plots):
Warning, the name changecoords has been redefined
> plot({cos(2*t),cos(5*t),2*cos(2*t) +
  3*cos(5*t)},t=0..12,color=black,title=
  'combined period is 2*Pi!');
```



Case 1b (interesting) $|\omega-\omega_0|$ is small.

write $\left.\begin{array}{l}\bar\omega = \frac{\omega+\omega_0}{2}\\ \delta = \frac{\omega-\omega_0}{2}\end{array}\right\}$ so $\begin{array}{l}\omega = \bar\omega+\delta\\ \omega_0 = \bar\omega - \delta\end{array}$

From addition angle,

$\cos((\bar\omega+\delta)t) - \cos((\bar\omega-\delta)t)$

$= \cancel{\cos\bar\omega t\cos\delta t} - \sin\bar\omega t\sin\delta t$
$-(\cancel{\cos\bar\omega t\cos\delta t} + \sin\bar\omega t\sin\delta t)$

$= -(2\sin\delta t)(\sin\bar\omega t)$

slowing varying "amplitude" function with period $|2\pi/\delta|$

oscillates with $\bar\omega \approx \omega \approx \omega_0$

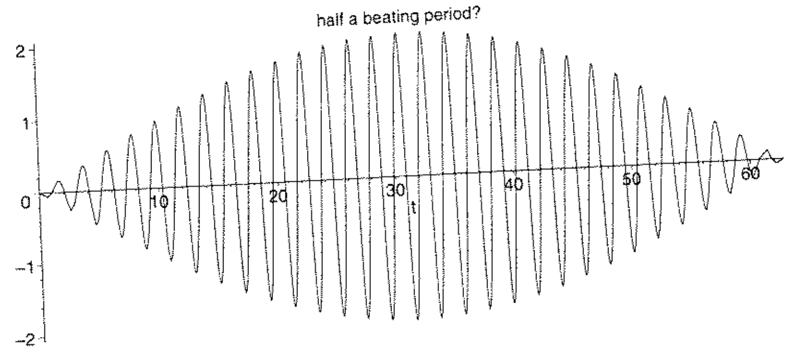
Rewrite page 1 sol'n:

$$\boxed{x(t) = \frac{F_0}{m(\omega_0^2-\omega^2)}(-2)\sin\left(\frac{\omega-\omega_0}{2}t\right)\sin\left(\frac{\omega+\omega_0}{2}t\right) + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

This is called "beating", see example 2. p 212 (or Maple)
- Very important to musicians
- the transition to resonance!

Beating

```
> plot(cos(31/10*t)-cos(3*t),
    t=0..62.8, color=black,title='half a beating period?');
```




```
>
```

<u>Case 2</u> $c=0,\ \omega=\omega_0$ ("resonance")

$$mx'' + kx = F_0 \cos\omega_0 t$$

Find A and B so that

$+\ k[\quad x_p(t) = (A\cos\omega_0 t + B\sin\omega_0 t)\,t$

$0[\quad x_p'(t) =$

$+\ m[\quad x_p''(t) =$

$Lx_p''(t) =$

<u>ans</u> $x_p(t) = \frac{F_0}{2m\omega_0}\, t\sin\omega_0 t$

IVP soltn: $\boxed{x(t) = \frac{F_0}{2m\omega_0}\, t\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$

For $0 \le t \le M$ this formula can also be obtained from page 2 formula by letting $\omega \to \omega_0$!

Example:

$F_0 = 5$

$m = .1$

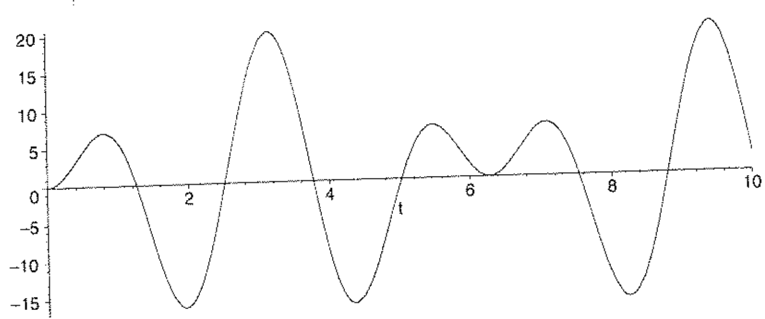
$\omega_0 = 3$

$\omega = 2$: $\frac{F_0}{m}\frac{1}{\omega_0^2-\omega^2}\left[\cos\omega t - \cos\omega_0 t\right]$

$= \frac{50}{5}\left(\cos 2t - \cos 3t\right)$

(page 1 formula)

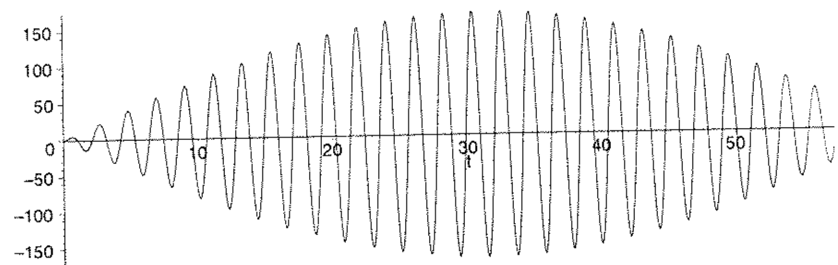
```
> plot(10*(cos(2*t)-cos(3*t)),t=0..10,color=black);
```



$\omega = 2.9$ $\quad x(t) = \frac{50(-2)}{(9-(2.9)^2)}\sin(-.05t)\sin(2.95t)$

(page 2 formula)

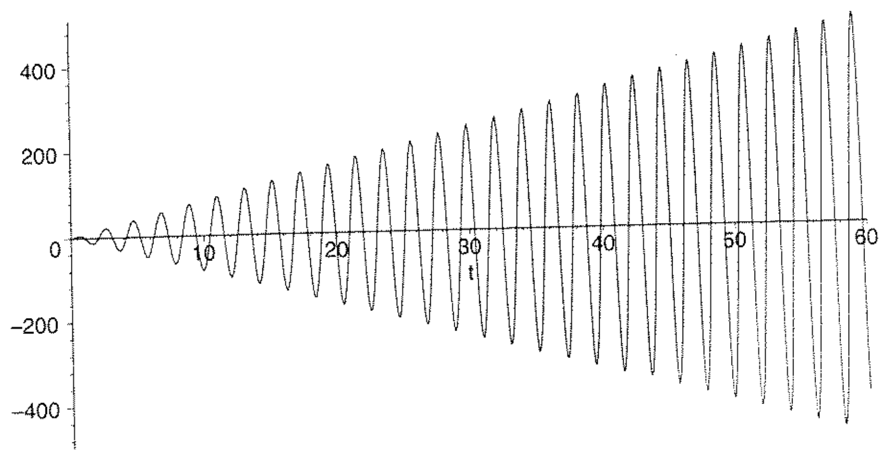
```
> plot(-100/(9-2.9^2)*sin(-.05*t)*sin(2.95*t),t=0..60,
  numpoints=200,color=black);
```



$\omega = 3$ $\quad x(t) = \frac{25}{3} t\sin 3t$

(page 3 formula)

```
> plot(25/3*t*sin(3*t),t=0..60,numpoints=200,color=black);
```



Tues: Case 3 $c \neq 0$!