Math 2280-1
Monday Oct 6

HW for 10/20 (1)
(next week is semester break!)
3.5 3, 4, 17, 19, 36, 37, 43, 49, 50, 51, 64
3.6 4, 5, 7, 13, 16, 21, 22
4.1 1, 8, 11, 15, 16, 21a, 24, 26

§3.5 - we already did "undetermined coeff's" for $Ly = f$, L const coef n^{th} order linear

another way:

variation of parameters : If, for $L(y) := y^{(n)} + a_{n-1}(x) y^{(n-1)} + \ldots + a_1(x) y' + a_0(x) y$

you already have a basis for $\ker(L)$, i.e. the sol'n set to $L(y) = 0$, there is a formula to find a particular sol'n y_P to

$$L(y_P) = f.$$

(this will work for any right hand side, not just the special ones we considered for "undetermined coeff".)

derivation: Let $\{y_1, y_2, \ldots y_n\}$ a basis for $\ker(L)$

we search for y_P expressed as

$u_j = u_j(x)$ fcns! (hence the name of the method).

$$\begin{aligned}
a_0 (\quad & y_P(x) = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n \\
a_1 (\quad & y_P' = u_1 y_1' + u_2 y_2' + \cdots + u_n y_n' + \underbrace{u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n}_{\text{set } \equiv 0} \\
+ \ a_2 (\quad & y_P'' = u_1 y_1'' + u_2 y_2'' + \cdots + u_n y_n'' + \underbrace{u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n'}_{\text{set } \equiv 0} \\
& \vdots \\
+ \ a_{n-1} (\quad & \\
+ \ 1 (\quad & y_P^{(n)} = u_1 y_1^{(n)} + u_2 y_2^{(n)} + \cdots + u_n y_n^{(n)} + \underbrace{u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)}}_{\text{set } \equiv f}
\end{aligned}$$

$$\Longrightarrow \quad L(y_P) = \underbrace{u_1 L(y_1) + \cdots + u_n L(y_n)}_{=0!} + f$$

conditions in matrix form

$$\underbrace{\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{bmatrix}}_{W(y_1, \ldots y_n).} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{bmatrix}$$

$$\boxed{\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = [W]^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f \end{bmatrix}}$$

then find choices of each $u_j(x)$ by antidifferentiation,

& get

$$y_P = u_1 y_1 + \cdots + u_n y_n \ !$$

Theorem: since $\{y_1, \ldots y_n\}$ is a basis we can solve all IVP's at each $x \Rightarrow [W]$ has rank n at each $x \Rightarrow W^{-1}$ exists at each x!

Example

forced spring, $mx'' + cx' + kx = F(t)$; e.g.

$$x''(t) + 16x(t) = \sin 4t$$

what would your guess be for $x_p(t)$ using guessing? ← work problem this way too!

$$L(x) := x'' + 16x$$

$$\ker L = \operatorname{span}\{\underset{\substack{\uparrow \\ x_1}}{\cos 4t}, \underset{\substack{\uparrow \\ x_2}}{\sin 4t}\}.$$

var par formula from page 1:

$$\begin{bmatrix} \cos 4t & \sin 4t \\ -4\sin 4t & 4\cos 4t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \sin 4t \end{bmatrix}.$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4\cos 4t & -\sin 4t \\ 4\sin 4t & \cos 4t \end{bmatrix}\begin{bmatrix} 0 \\ \sin 4t \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -\sin^2 4t \\ \sin 4t \cos 4t \end{bmatrix}$$

$$u_1' = \frac{-\sin^2 4t}{4} = \frac{\cos 8t - 1}{8} \qquad \text{(double/half angle formula)}$$

$$u_2' = \frac{1}{4}\sin 4t \cos 4t$$

choose $u_1 = \dfrac{\sin 8t}{64} - \dfrac{t}{8}$

$u_2 = \dfrac{\sin^2 4t}{16}$

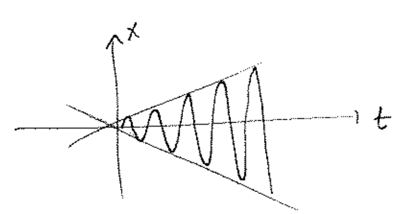
$$x_p = u_1x_1 + u_2x_2 = \left(\frac{\sin 8t}{64} - \frac{t}{8}\right)(\cos 4t) + \frac{\sin^2 4t}{32}(\sin 4t)$$

$$= \frac{2}{64}\sin 4t \cos 4t \cos 4t - \frac{t}{8}\cos 4t + \frac{1}{32}(1-\cos^2 4t)(\sin 4t)$$

$$= -\frac{t}{8}\cos 4t + \underbrace{\frac{1}{32}\sin 4t}_{\text{solves } L(x)=0}$$

so may take

$$\boxed{x_p(t) = -\frac{t}{8}\cos 4t}$$



resonance!