Math 2280-1

Wednesday October 29

§ 5.4 : 1st order systems $\frac{d\vec{x}}{dt} = A\vec{x}$

when A is not diagonalizable

Do experiment first!
(pages 6-7).
then begin today's notes
(and finish them on Friday)

Recall (2270),

for any $A_{n\times n}$,

$$|A-\lambda I| = (-1)^n (\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\cdots(\lambda-\lambda_m)^{k_m} \qquad \lambda_i \in \mathbb{C} \quad 1 \le i \le m$$

λ_i distinct

$$k_1+k_2+\cdots+k_m = n$$

$E_{\lambda_i} = \lambda_i$-eigenspace

- $\dim(E_{\lambda_i}) \le k_i$ always holds (geometric multiplicity $\le$ algebraic mult.)
- A is diagonalizable (has an $\mathbb{R}^n$ eigenbasis) iff each $\dim(E_{\lambda_i}) = k_i$

and in this (diagonalizable) case, we can find the general soltn to $\frac{d\vec{x}}{dt} = A\vec{x}$ as linear combos of $\{e^{\lambda_1 t}\vec{v}_1, \ldots e^{\lambda_m t}\vec{v}_n\}$.

($\exists$ analog for complex evals & evects)

example 1 p 333

$$\vec{x}' = \begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix}\vec{x}$$

$$|A-\lambda I| = -(\lambda-5)(\lambda-3)^2$$

$\lambda = 5$

$$\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda = 3$:

$$\left[\begin{array}{ccc|c} 6 & 4 & 0 & 0 \\ -6 & -4 & 0 & 0 \\ 6 & 4 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$
$x_2 = s$
$x_1 = -\frac{2}{3}s$

$$\vec{x} = s\begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\vec{x}_H(t) = c_1 e^{5t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + c_3 e^{3t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

all is well!
(even though $\lambda = 3$ had algebraic mult = 2).

But, how about

$$\vec{x}' = \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -9 & 9 \end{bmatrix}}_{B} \vec{x}$$

$$|B - \lambda I| = \begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & -3-\lambda & 4 \\ 0 & -9 & 9-\lambda \end{vmatrix} = (5-\lambda)\left[\lambda^2 - 6\lambda \overbrace{-27+36}^{+9}\right]$$
$$= (5-\lambda)(\lambda-3)^2$$

same as example 1.

$\lambda = 5$ $\qquad$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = 3$ $\qquad$ $\vec{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ $\quad$ (Maple)

$\lambda = 3$ eigenspace is "defective"

$$\left\{ e^{5t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e^{3t}\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, ?? \right\}$$

discussion:

Let $A\vec{v} = \lambda\vec{v}$, but λ-eigenspace "defective".

$e^{\lambda t}\vec{v}$ solves $\frac{d\vec{x}}{dt} = A\vec{v}$

does $\vec{z}(t) = te^{\lambda t}\vec{v}$?

$$\frac{d\vec{z}}{dt} = e^{\lambda t}\vec{v}\left[1 + \lambda t\right]$$
$$A\vec{z} = te^{\lambda t}A\vec{v} = te^{\lambda t}\lambda\vec{v}$$
$$\frac{d\vec{z}}{dt} - A\vec{z} = e^{\lambda t}\vec{v}$$

so $\vec{z}(t)$ fails.

how about

$$\vec{z}(t) = e^{\lambda t}(t\vec{v} + \vec{w}) \ ?$$
$$\vec{z}'(t) = e^{\lambda t}\left[\lambda t\vec{v} + \lambda\vec{w} + \vec{v}\right]$$
$$A\vec{z} = e^{\lambda t}\left[t\lambda\vec{v} + A\vec{w}\right]$$

success iff $\boxed{A\vec{w} = \lambda\vec{w} + \vec{v}}$

$\boxed{(A - \lambda I)\vec{w} = \vec{v}}$

in our example: $\left(\lambda = 3,\ \vec{v} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}\right)$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & -6 & 4 & 2 \\ 0 & -9 & 6 & 3 \end{array}\right]$$

R2/-2
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$w_3 = t$$
$$w_2 = -\tfrac{1}{3} + \tfrac{2}{3}t$$
$$w_1 = 0$$

$$\vec{w} = \begin{bmatrix} 0 \\ -1/3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 2/3 \\ 1 \end{bmatrix}$$

mults of $\vec{v}$

can take $\vec{w} = \begin{bmatrix} 0 \\ -1/3 \\ 0 \end{bmatrix}$

get soln

$$?? = \vec{z}(t) = e^{\lambda t}\left\{ t\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/3 \\ 0 \end{bmatrix} \right\}$$

this gives a 3rd lin ind sol'n!!

<u>Theorems</u>: Let $A_{n\times n}$ as on page 1, with

$$|A-\lambda I| = (-1)^n (\lambda-\lambda_1)^{k_1} (\lambda-\lambda_2)^{k_2} \cdots (\lambda-\lambda_m)^{k_m}$$

$k_1 + \dots + k_m = n$
$\lambda_i \in \mathbb{C} \quad 1 \le i \le m$
λ_i distinct

(1) The <u>generalized</u> λ_i-eigenspace is defined to be

$$\ker\left(A - \lambda_i I\right)^{k_i}$$

it <u>is</u> always k_i dimensional (in $\mathbb{C}^n$)

(2) By amalgamating bases of the generalized eigenspaces you get a basis for $\mathbb{C}^n$

(3) The generalized λ_i-eigenbases can be chosen to be made of one or more "chains" $\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_k\}$ where for $\lambda = \lambda_i$

$$\begin{aligned}
(A-\lambda I)\vec{u}_1 &= \vec{0} \\
(A-\lambda I)\vec{u}_2 &= \vec{u}_1 \\
&\vdots \\
(A-\lambda I)\vec{u}_k &= \vec{u}_{k-1}
\end{aligned}$$

($\vec{u}_1$ <u>is</u> an eigenvector)

(on page 2, $(\vec{v}, \vec{w})$ was a chain of length 2)

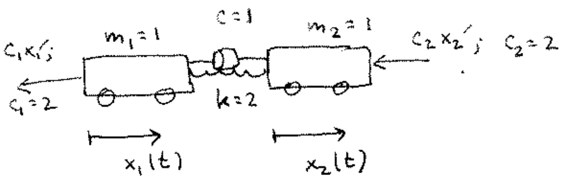
(4) Each chain $\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_k\}$ yields k lin. ind. sol'ns to $\frac{d\vec{x}}{dt} = A\vec{x}$:

$$\begin{aligned}
\vec{x}_1(t) &= \vec{u}_1 e^{\lambda t} \\
\vec{x}_2(t) &= (\vec{u}_1 t + \vec{u}_2) e^{\lambda t} \\
\vec{x}_3(t) &= \left(\vec{u}_1 \tfrac{1}{2} t^2 + \vec{u}_2 t + \vec{u}_3\right) e^{\lambda t} \\
&\vdots \\
\vec{x}_k(t) &= \left(\vec{u}_1 \tfrac{1}{k!} t^k + \dots + \vec{u}_{k-1} t + \vec{u}_k\right) e^{\lambda t}
\end{aligned}$$

amalgamating these solutions yields a basis of sol'ns to $\frac{d\vec{x}}{dt} = A\vec{x}$!!!

<u>Proof</u> we will discuss aspects of the general proof during following lectures

Example 6 p. 343



$$m_1 x_1'' = +k(x_2 - x_1) - c_1 x_1' + c(x_2' - x_1')$$
$$m_2 x_2'' = -k(x_2 - x_1) - c_2 x_2' - c(x_2' - x_1')$$

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

cannot use §5.3 (no damping.)
Must convert to 1st order system

$x_1 = x_1$
$x_2 = x_2$
$x_3 = x_1'$
$x_4 = x_2'$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Maple to the rescue! (See next page).

$$\begin{bmatrix} x_1 \\ x_2 \\ x_1' \\ x_2' \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + e^{-2t}\left[c_2 \begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix} + c_4 \overbrace{\left[t \begin{bmatrix} -1 \\ 1 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right]}^{t\vec{u}_1 + \vec{u}_2} \right]$$

explain.

nice physically; the sum of MAPLE's 2 $\lambda = -2$ eigenvects

the difference of Maple's 2 $\lambda = -2$ eigenvects. also nice physical interpretation

```
> with(linalg):
Warning, the protected names norm and trace have been redefined and unprotected
> A:=matrix(4,4,[0,0,1,0,0,0,0,1,-2,2,-3,1,2,-2,1,-3]);
```

$$A := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{bmatrix}$$

```
> eigenvects(A);
```

$$[0, 1, \{[1, 1, 0, 0]\}], [-2, 3, \{[1, 0, -2, 0], [0, 1, 0, -2]\}]$$

the lambda = -2 eigenvalue is defective!!!!

```
> Iden:=array(1..4,1..4,identity);
```

$$Iden := \text{array}(identity, 1 .. 4, 1 .. 4, [\,])$$

```
> evalm(Iden);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
> B:=evalm(A+2*Iden); #the matrix A-lambda*I
```

$$B := \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{bmatrix}$$

```
> kernel(B^3); #this is the generalized lambda=-2
   #eigenspace.  It is indeed 3-dimensional
```

$$\{[1, 0, -2, 0], [0, 1, -2, 0], [0, 0, -1, 1]\}$$

```
> kernel(B^2);  #I will need a chain of length 2,
   #there won't be a chain of length 3 since
   #the actual eigenspace is 2-d
```

$$\{[1, 0, -2, 0], [0, 1, -2, 0], [0, 0, -1, 1]\}$$

```
> u2:=([0, 0, -1, 1]); #for this chain don't start
    #with an actual eigenvector.  I like this
    #vector because of its physical meaning in
    #the train problem!
```

$$u2 := [0, 0, -1, 1]$$

```
> u1:=evalm(B&*u2);
```

$$u1 := [-1, 1, 2, -2]$$

```
> evalm(B&*u1);  #should be zero
```

$$[0, 0, 0, 0]$$

Calculations for a 2 mass- 3 spring system

Math 2280-1,
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The two mass, three spring system.

Data: Each ball mass is 50 grams. Each spring mass is 6 grams. (Remember, and this is a defect, our model assumes massless springs.) The springs are "identical", and an extra mass of 50 grams stretches the spring 18.0 centimeters from equilibrium. (We can recheck this.). Thus the spring constant is given by

```
> solve(k*.18=.05*9.8,k);
```

$$2.722222222$$

Let's time the two natural periods (which we discuss below):
(For the fast one, in my office, I got 50 cycles in about 25.14 seconds. (Hard to count this one!) For the slow one I got 20 cycles in about 18.03 seconds. What do we get in class?)

Here's the model:

```
> with(linalg):
> A:=matrix(2,2,[-2*k/m, k/m,k/m,-2*k/m]);
   #this should be the "A" matrix you get for
   #our two-mass, three-spring system.
```

$$A := \begin{bmatrix} -\frac{2\,k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2\,k}{m} \end{bmatrix}$$

```
> eigenvects(A);
```

$$\left[-\frac{k}{m}, 1, \{[1, 1]\}\right], \left[-\frac{3\,k}{m}, 1, \{[-1, 1]\}\right]$$

```
>
```

Predict the two natural periods from the model:

ANSWER: If you do the model correctly and our data is correct, you will come up with natural periods of .49 and .85 seconds. I predict that the real natural periods are a little longer. What happened?

EXPLANATION: The springs actually have mass, equal to 6 grams each. This is almost on the same order of magnitude as the ball masses, and causes the actual experiment to run more slowly than our model predicts. In order to be more accurate the total energy of our model must account for the kinetic energy of the springs. You actually have the tools to model this more-complicated situation, using the ideas of total energy discussed in section 5.6, and a "little" Calculus. You can carry out this analysis, like I sketched for the single mass, single spring oscillator (sept30.pdf), assuming that the spring velocity at a point on the spring linearly interpolates the velocity of the wall and mass (or mass and mass) which bounds it. It turns out that this gives an A-matrix the same eigenvectors, but different eigenvalues, namely

$$\lambda_1 = -\frac{k}{m + m_s}$$

$$\lambda_2 = -\frac{9\,k}{m_s + 3\,m}$$

(Hints: the "M" matrix is not diagonal, the "K" matrix is the same, and you can ignore PE from gravity in the "total energy" formulation, because gravity just resets the equilibrium positions and doesn't effect the vibrations).

If you use these values, then you get period predictions

```
> m:=.05:
  ms:=.006;
  k:=2.722;
  Omega1:=sqrt(k/(m+ms));
  Omega2:=sqrt(9*k/(ms+3*m));
  T1:=evalf(2*Pi/Omega1);
  T2:=evalf(2*Pi/Omega2);
```

$$ms := 0.006$$

$$k := 2.722$$

$$\Omega 1 := 6.971882304$$

$$\Omega 2 := 12.53149878$$

$$T1 := 0.9012179260$$

$$T2 := 0.5013913672$$

```
>
```

of .90 and .50 seconds per cycle. Is that closer?

Challenge: If you can construct (and explain to me in my office, along with a written explanation) a correct derivation of the eigenvalues /eigenvectors I claim above, by taking the spring masses into account, then you can either substitute your derivation for the section 5.3 Maple exploration in this week's homework, or get 10 bonus points on the next midterm. This is a challenging challenge, but it's definitely doable.