Math 2280-1

Tuesday October 28

§5.3 Mass-spring systems.

Work example 1., page 321, using Monday's class notes

Here's a Maple check of part of our work:

```
> with(linalg):with(plots):with(DEtools): #tools for project
> M:=matrix([[2,0],[0,1]]);
  K:=matrix([[-150,50],[50,-50]]);
  A:=evalm(inverse(M)&*K);
```

$$M := \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K := \begin{bmatrix} -150 & 50 \\ 50 & -50 \end{bmatrix}$$

$$A := \begin{bmatrix} -75 & 25 \\ 50 & -50 \end{bmatrix}$$

```
> eigenvectors(A);
```

[-100, 1, {[-1, 1]}], [-25, 1, {[1, 2]}]

Now do forced oscillations:

$$M\vec{x}'' = K\vec{x} + \vec{b}\cos\omega t$$

or $\boxed{\vec{x}'' = A\vec{x} + \vec{F}_0\cos\omega t}$ $\qquad \vec{F}_0 = M^{-1}\vec{b}$.

try $\vec{x}_p = \vec{c}\cos\omega t$ [assuming ω is <u>not</u> one of the natural angular frequencies]

$$\vec{x}_p{}'' = \vec{c}(-\omega^2\cos\omega t)$$

$$\text{want} = A\vec{c}\cos\omega t + \vec{F}_0\cos\omega t$$
$$= (A\vec{c} + \vec{F}_0)\cos\omega t$$

$$-\omega^2\vec{c} = A\vec{c} + \vec{F}_0$$
$$-\vec{F}_0 = A\vec{c} + \omega^2 I\vec{c}$$
$$-\vec{F}_0 = (A + \omega^2 I)\vec{c}$$
$$\boxed{\vec{c} = (A + \omega^2 I)^{-1}(-\vec{F}_0)}$$

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trouble when $\omega^2 = -\lambda$

since $A - \lambda I$ is sing!

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$$\vec{x}'' = \begin{bmatrix} -75 & 25 \\ 50 & -50 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 50 \end{bmatrix} \cos\omega t$$

$$\vec{x}_p = \vec{c}\cos\omega t$$

$$-\omega^2 \vec{c}\cos\omega t = \begin{bmatrix} -75 & 25 \\ 50 & -50 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \cos\omega t + \begin{bmatrix} 0 \\ 50 \end{bmatrix} \cos\omega t$$

$$\| $$

$$\begin{bmatrix} -\omega^2 c_1 \\ -\omega^2 c_2 \end{bmatrix} \cos\omega t$$

$$\begin{bmatrix} 0 \\ -50 \end{bmatrix} = \begin{bmatrix} \omega^2-75 & 25 \\ 50 & \omega^2-50 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$c_1 = \frac{\begin{vmatrix} 0 & 25 \\ -50 & \omega^2-50 \end{vmatrix}}{(\omega^2-75)(\omega^2-50) - 50\cdot 25}$$

$$\boxed{c_1 = \frac{1250}{(\omega^2-25)(\omega^2-100)} \qquad c_2 = \frac{50(\omega^2-75)}{(\omega^2-25)(\omega^2-100)}}$$

Interpret!?

Maple version

```
> F0:=evalm(inverse(M)&*vector([0,50]));
      #The F0 in the normalized equation (32), page 327
  Iden:=array(1..2,1..2,identity);
      #the 2 by 2 identity matrix
  Aleft:=omega->evalm(A + omega^2*Iden);
      #the matrix function multiplying
      #c on the left side of (32)
  c:=omega->evalm(-inverse(Aleft(omega))&*F0);
      #the solution vector c(omega) to (32),
      #obtained by multiplying both sides of equation
      #(32) on the left, by the inverse to Aleft
```

$$F0 := [0, 50]$$

$$Iden := \mathrm{array}(identity, 1\,..\,2, 1\,..\,2, [\,])$$

$$Aleft := \omega \to \mathrm{evalm}(A + \omega^2\, Iden)$$

$$c := \omega \to \mathrm{evalm}(-\text{`\&*`}(\mathrm{inverse}(\mathrm{Aleft}(\omega)), F0))$$

```
> c(omega); #see equation (35) page 323
```

$$\left[\frac{1250}{2500 - 125\,\omega^2 + \omega^4}, -\frac{50\,(-75+\omega^2)}{2500 - 125\,\omega^2 + \omega^4} \right]$$

The vector c(w) above, times the oscillation cos(wt), is a particular solution to the forced oscillation problem we are considering. If we assume that our actual problem has a small amount of damping, then we expect that this particular solution is very close to the steady periodic solution to the damped problem. See the discussion on page 327. We can study resonance phenomena for these slightly damped problems by plotting the maximum amplitude of the steady state solutions to the undamped problems. That would be the maximum absolute value of c1 and c2 above. Use the Maple command ''norm'' to measure this maximum amplitude:

```
> norm(c(omega));
```

$$\max\left(\frac{1250}{\left|2500-125\,\omega^2+\omega^4\right|},\ 50\left|\frac{-75+\omega^2}{2500-125\,\omega^2+\omega^4}\right|\right)$$

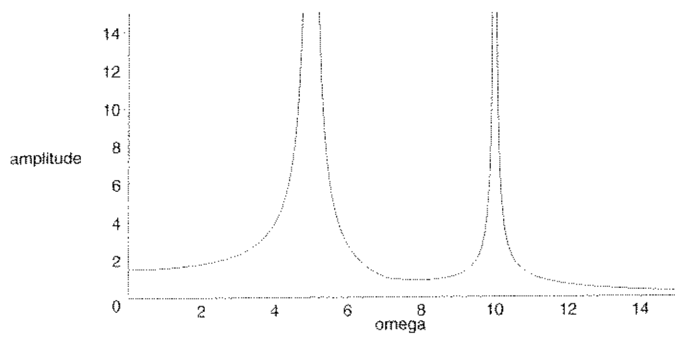
Another way to measure the size of c(omega) is to take its Euclidean magnitude, which is the command

```
> norm(c(omega),2);
```

$$50\sqrt{\frac{625}{\left|2500-125\,\omega^2+\omega^4\right|^2}+\left|\frac{-75+\omega^2}{2500-125\,\omega^2+\omega^4}\right|^2}$$

(You will use the first command in the Earthquake project, which perhaps makes the most sense since it will be measuring the maximum amplitude that any floor oscillates.) The following picture illustrates that the maximum amplitude of the particular solution blows up when omega is near the two natural angular frequences. Thus, in the slightly damped problem, one would experience practical resonance in the steady periodic solution.

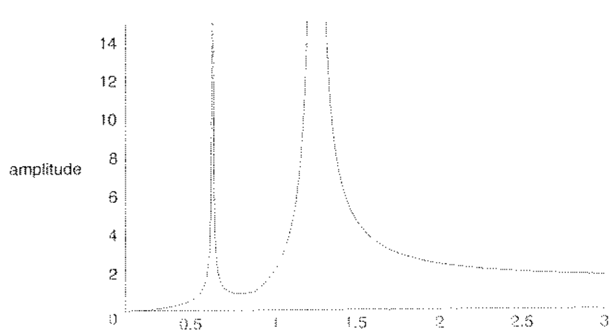
```
> plot(norm(c(omega)),omega=0..15,amplitude=0..15,
          numpoints=200,color='black');
```



This is qualitatively the picture on page 327, figure 5.3.10, although they plotted the Euclidean magnitude of c(omega) rather than the maximum amplitude. Notice how we get Maple to label the axes as desired

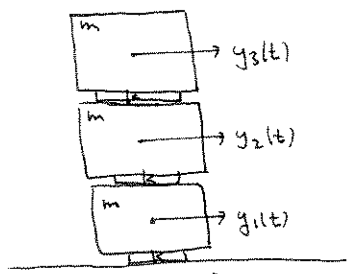
We can get a plot of resonance as a function of period by recalling that 2*Pi/T=omega:

```
> plot(norm(c(2*Pi/period)),period=0.1..3,amplitude=0..15,
  numpoints=200,color='black');
```



Earthquake Project Comments

consider a 3 story building (in project is 7 stories) modeled as a mass-spring model, with each story having mass m, and floor junctions modeled as springs with constants k:



ground $y_0(t)$ → transverse (shaking) wave $E\cos\omega t$

$y_0(t) = E\cos\omega t$ so

$$y_0'' = -E\omega^2\cos\omega t$$
$$m y_1'' = -k(y_1 - y_0) + k(y_1 - y_2)$$
$$m y_2'' = -k(y_2 - y_1) - k(y_2 - y_3)$$
$$m y_3'' = -k(y_3 - y_2)$$

reduction of fns

let

$$x_0 = y_0 - E\cos\omega t \equiv 0$$
$$x_1 = y_1 - E\cos\omega t$$
$$x_2 = y_2 - E\cos\omega t$$
$$x_3 = y_3 - E\cos\omega t$$

$$x_0 \equiv 0$$
$$m x_1'' = m y_1'' + mE\omega^2\cos\omega t$$
$$m x_2'' = m y_2'' + mE\omega^2\cos\omega t$$
$$m x_3'' = m y_3'' + mE\omega^2\cos\omega t$$

so

$$m x_1'' = -k(x_1 - 0) - k(x_1 - x_2) + mE\omega^2\cos\omega t$$
$$m x_2'' = -k(x_2 - x_1) - k(x_2 - x_3) + mE\omega^2\cos\omega t$$
$$m x_3'' = -k(x_3 - x_2) + mE\omega^2\cos\omega t$$

so the x's are the displacements from equilibrium as measured by an observer on the ground.

Notice $y_i - y_j = x_i - x_j$

$$\boxed{\begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{bmatrix} = \begin{bmatrix} -2k/m & k/m & 0 \\ k/m & -2k/m & k/m \\ 0 & k/m & -k/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + E\omega^2\cos\omega t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

it's as if all floors are being forced – book calls this inertial forcing due to frame of reference