Math 2280-1

Friday Oct 24 §5.1-5.2 ~ actually, I claim we did §5.1 on Tuesday!!

Bring Tuesday notes, so we can make sure we understand the theory in §5.1 about solutions to $\vec{x}'(t) - A(t)\vec{x}(t) = \vec{f}(t)$

i.e. $\vec{x} = \vec{x}_p + \vec{x}_H$

$\dim \{\vec{x}_H\} = n$, etc.

Example 1 p.307 (§5.2)

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- What is the dimension of the sol'n space to this system of DE's?

- Find a basis for sol'n space of the form $\{ e^{\lambda_1 t}\vec{v}_1, e^{\lambda_2 t}\vec{v}_2 \}$.
  (recall, for $\vec{x}' = A\vec{x}$, $\vec{x} = e^{\lambda t}\vec{v}$ is a sol'n iff $A\vec{v} = \lambda\vec{v}$).

- What is the Wronskian for your basis above?

Example 1 cont'd

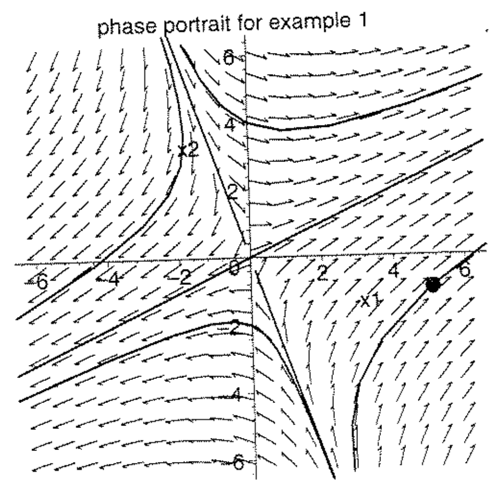
Solve

$$\begin{cases} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \end{cases}$$

ans

$$\vec{x}(t) = e^{-2t}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + 2e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

```
>
> with(DEtools):
> phaseportrait([diff(x1(t),t)=4*x1(t)+2*x2(t),
  diff(x2(t),t)=3*x1(t)-x2(t)],
  [x1(t),x2(t)],t=-1..1,
  {[x1(0)=5,x2(0)=-1],[x1(0)=0,x2(0)=4],
  [x1(0)=2,x2(0)=1],[x1(0)=-1,x2(0)=3],
  [x1(0)=-2,x2(0)=-1],[x1(0)=1,x2(0)=-3],
  [x1(0)=-3,x2(0)=1],[x1(0)=0,x2(0)=-2]
  },x1=-6..6,x2=-6..6,
  color=black,linecolor=black,
  title=`phase portrait for example 1`);
```




<u>Theorem 1</u> If $A_{n\times n}$ is diagonalizable, i.e. has an $\mathbb{R}^n$ basis of eigenvectors $\{\vec{v}_1, \vec{v}_2, .. \vec{v}_n\}$ (with corresponding evals $\lambda_1, .. \lambda_n$), then

$$\left\{ e^{\lambda_1 t}\vec{v}_1, \dots e^{\lambda_n t}\vec{v}_n \right\} \text{ is a basis of soltns to } \frac{d\vec{x}}{dt} = A\vec{x}$$

this is guaranteed to happen if $\lambda_1, \lambda_2, .. \lambda_n$ are distinct

<u>proof</u>: 2270 for the last statement, i.e. $\lambda_1, .. \lambda_n$ distinct $\Rightarrow \{\vec{v}_1, .. \vec{v}_n\}$ l.i. $\Rightarrow \mathbb{R}^n$ Basis

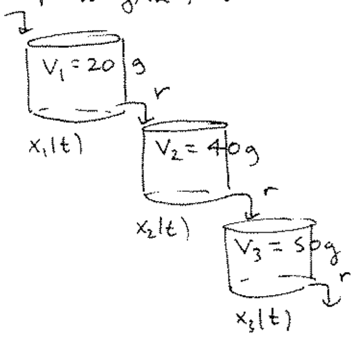
(we can recall the proof!)

(there's a cool one using Vandermonde det)

$\Rightarrow e^{\lambda_1 t}\vec{v}_1, .. e^{\lambda_n t}\vec{v}_n$ l.i. ∎

example 2 p. 309

$r = 10$ g/m, $c_i = 0$




$$\frac{dx_1}{dt} = (-10)\frac{x_1}{20} = -.5x_1$$

$$\frac{dx_2}{dt} = 10\frac{x_1}{20} - 10\frac{x_2}{40}$$

$$\frac{dx_3}{dt} = 10\frac{x_2}{40} - 10\frac{x_3}{50}$$

given $x_1(0) = 15$ (lb)
$x_2(0) = 0$
$x_3(0) = 0$

Find $\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -.5 & 0 & 0 \\ .5 & -.25 & 0 \\ 0 & .25 & -.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Use MAPLE!!
(knowing you could work by hand)

```
> with(linalg):
> A:=matrix(3,3,[-.5,0,0,.5,-.25,0,0,.25,-.2]);
```

$$A := \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{bmatrix}$$

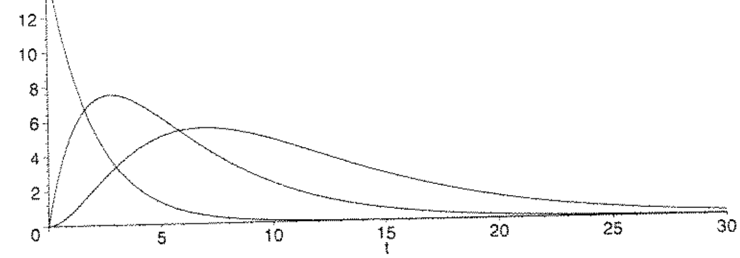
```
> eigenvectors(A);
```

[-0.5, 1, {[1, -2.000000000, 1.666666667]}], [-0.25, 1, {[0, 1, -5.000000]}],
[-0.2, 1, {[0, 0, 1]}]

So $\vec{x}(t) = c_1 e^{-.5t}\begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} + c_2 e^{-.25t}\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} + c_3 e^{-.2t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\vec{x}(0) = \begin{bmatrix} 3 & 0 & 0 \\ -6 & 1 & 0 \\ 5 & -5 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} c_1 = 5 \\ c_2 = 30 \\ c_3 = 125 \end{matrix} \Rightarrow \vec{x}(t) = 5e^{-.5t}\begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} + 30e^{-.25t}\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} + 125e^{-.2t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

```
> plot({15*exp(-.5*t),-30*exp(-.5*t)+30*exp(-.25*t),
  25*exp(-.5*t)-150*exp(-.25*t)+125*exp(-.2*t)},t=0..30,
  color=black,title=`amounts in each tank`);
```



**Example 3: Glucose-insulin model** (adapted from a discussion on page 340-341 of the text "Linear Algebra with Applications," by Otto Bretscher)

Let G(t) be the excess glucose concentration (mg of G per 100 ml of blood, say) in someone's blood, at time t hours. Excess means we are keeping track of the difference between current and equilibrium ("fasting") concentrations. Similarly, Let H(t) be the excess insulin concentration at time t. When blood levels of glucose rise, say as food is digested, the pancreas reacts by secreting insulin in order to utilize the glucose. Researchers have developed mathematical models for the glucose regulatory system. Here is a simplified (linearized) version of one such model, with particular representative matrix coefficients. It would be meant to apply between meals, when no additional glucose is being added to the system:

$$\begin{bmatrix} \frac{dG}{dt} \\ \frac{dH}{dt} \end{bmatrix} = \begin{bmatrix} -.1 & -.4 \\ .1 & -.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

**Explain (understand) the signs of the matrix coefficients:**

In Bretscher the topic at hand was **discrete dynamical systems** rather than **continuous** ones, and the model given was:

$$\begin{bmatrix} G(t+1) \\ H(t+1) \end{bmatrix} = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix} \begin{bmatrix} G(t) \\ H(t) \end{bmatrix}$$

Explain how I converted this discrete dynamical system into a related first order system of DE's! (Notice the matrices are NOT the same!)

Now let's solve the initial value problem, say right after a big meal, when

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

```
> restart:with(linalg):with(plots):
> A:=matrix(2,2,[-.1,-.4,.1,-.1]);
```

$$A := \begin{bmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{bmatrix}$$

```
> eigenvects(A);
```

$[-0.1 - 0.2000000000\,I,\ 1,\ \{[-2.000000000 - 0.\,I,\ 0. - 1.\,I]\}],$

$[-0.1 + 0.2000000000\,I,\ 1,\ \{[-2.000000000 + 0.\,I,\ 0. + 1.\,I]\}]$

Can you get the same eigenvalues and eigenvectors? Extract a basis for the solution space to his homogeneous system of differential equations from the eigenvector information above:

$\lambda = -.1 \mp .2i$

$\vec{v} = \begin{bmatrix} -2 \\ \pm i \end{bmatrix}$

(can you do this by hand?)

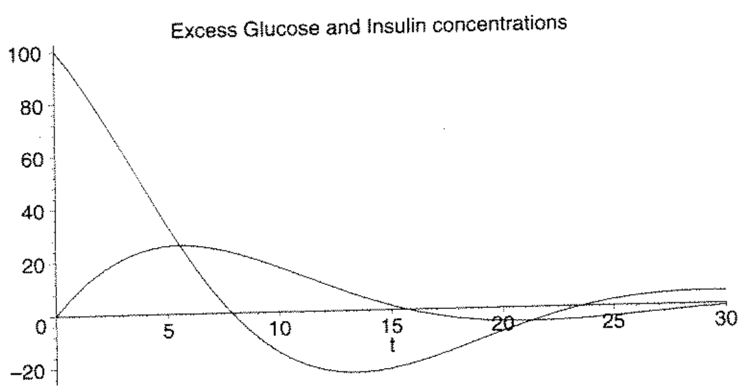
for $\lambda = -1 + .2i$

$e^{\lambda t} \vec{v} = e^{(-1+.2i)t} \begin{bmatrix} -2 \\ i \end{bmatrix} =$

Solve the initial value problem.

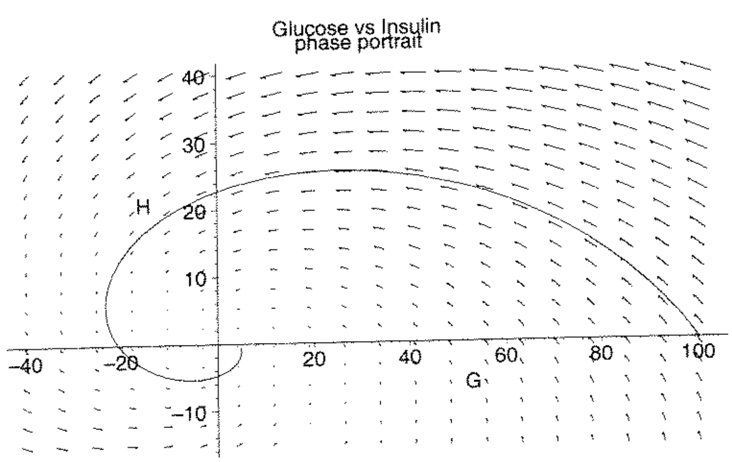
Here are some pictures to help understand what the model is predicting:
(1) Plots of glucose vs. insulin, at time t hours later:

```
> G:=t->100*exp(-.1*t)*cos(.2*t):
  H:=t->50*exp(-.1*t)*sin(.2*t):
> plot({G(t),H(t)},t=0..30,color=black,title=
  `Excess Glucose and Insulin concentrations`);
```




2) A phase portrait of the glucose-insulin system:

```
> pict1:=fieldplot([-.1*G-.4*H,.1*G-.1*H],G=-40..100,H=-15..40):
  soltn:=plot([G(t),H(t),t=0..30],color=black):
  display({pict1,soltn},title=`Glucose vs Insulin
  phase portrait`);
```



The class homework problem from Wednesday:

(A) Show that the $n^{th}$ order const coeff linear homog DE

(1) $y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y'(t) + a_0 y(t) = 0$

is equivalent to the $1^{st}$ order homogeneous system

(2) $$\begin{bmatrix} x_1(t) \\ x_2 \\ \vdots \\ x_n \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 - & 0 \\ \vdots \\ 0 & 0 & \cdots & \cdots & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \vec{x}'(t) = A\vec{x}$$

in the sense that if $y(t)$ solves (1), then $\begin{bmatrix} y \\ y' \\ y'' \\ \vdots \\ y^{(n-1)} \end{bmatrix}$ solves (2); and

if $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ solves (2), then $x_1(t)$ solves (1)

(B) Show that the eigenvalue characteristic polynomial $|A - rI|$ from the matrix $A$ in (2), is related to the characteristic poly $p(r)$ is (1), by

$$(-1)^n |A - rI| = p(r) = r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0.$$