Math 2280-1
Tuesday Oct 21

§4.1 $\exists!$ for 1^{st} order linear system IVP

HW for Monday 10/27

4.1 1, 8, 11, 15, 16, 21a, 24, 26

4.3 9 ← you may do this problem in groups of up to 2 people. You will need to write Runge-Kutta code!

5.1 11, 13, 18, 21, 22, 26, 31, 35

__Def__ A 1^{st} order system of DE's of the form

$$* \qquad \frac{d\vec{x}}{dt} - A(t)\vec{x} = \vec{f}(t) \qquad\qquad A(t) \text{ an } n\times n \text{ matrix}$$

is called a __linear__ 1^{st} order system.

If $\vec{f}(t) \equiv 0$ it is called __homogeneous__

Notice (check!)

$$L(\vec{x}(t)) := \frac{d\vec{x}}{dt} - A(t)\vec{x}(t)$$ is __linear__ (domain = continuously diff'ble vector valued fcns $\vec{x}(t)$; codomain = cont. vector valued fcns)

__Corollary__: The general sol'n to $*$ is

$$\vec{x}(t) = \vec{x}_H(t) + \vec{x}_p(t)$$

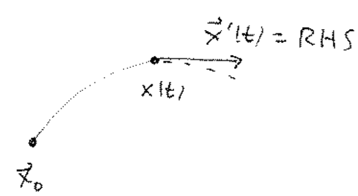
where $\vec{x}_p(t)$ is a particular sol'n to $*$ & $\vec{x}_H(t)$ is the general sol'n to the homogeneous DE

$$\frac{d\vec{x}}{dt} = A(t)\vec{x}.$$

__Theorem__ If $A(t)$ and $\vec{f}(t)$ are defined and continuous on an interval I, and $t_0 \in I$, then the IVP

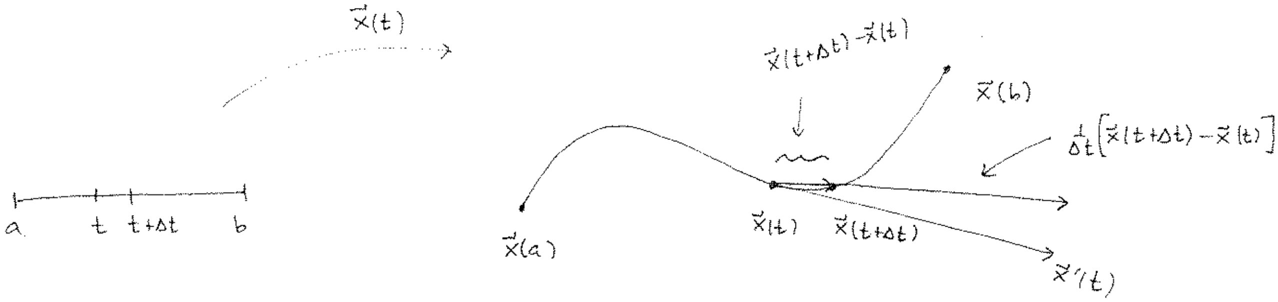
$$\begin{cases} \frac{d\vec{x}}{dt} = A(t)\vec{x} + \vec{f}(t) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

always has a unique sol'n on I



__proof__: omitted, but the idea is you know where to start at time t_0, and you know your tangent vector $\vec{x}'(t)$ if you know where you are and what time it is, so you know where to go. The linear nature of this DE turns out to guarantee local (in time) $\exists!$, and that the solution can't blow up in finite time, so that it exists $\forall t \in I$

Digression/review (2210 or 1260), about the meaning of $\vec{x}'(t)$



$$\underset{\text{domain}}{\vec{x}:[a,b]} \longrightarrow \underset{\text{codomain}}{\mathbb{R}^n}$$

$$\vec{x}'(t) := \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left(\vec{x}(t+\Delta t)-\vec{x}(t)\right)$$

$$= \lim_{\Delta t\to 0} \frac{1}{\Delta t}\begin{bmatrix} x_1(t+\Delta t)-x_1(t) \\ x_2(t+\Delta t)-x_2(t) \\ \vdots \\ x_n(t+\Delta t)-x_n(t) \end{bmatrix}$$

$$= \lim_{\Delta t\to 0} \begin{bmatrix} \frac{x_1(t+\Delta t)-x_1(t)}{\Delta t} \\ \vdots \\ \frac{x_n(t+\Delta t)-x_n(t)}{\Delta t} \end{bmatrix}$$

$$= \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}.$$

We called $\vec{x}'(t)$ the <u>tangent vector</u> or <u>velocity vector</u> (if $\vec{x}(t)$ was particle motion).

<u>Thus</u> the first order IVP solution to

$$\begin{cases} \frac{d\vec{x}}{dt} = \vec{F}(t,\vec{x}) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

can be interpreted as ~~starting~~ the particle motion starting at $\vec{x}_0$, for which the "velocity" vector is known, in terms of current position and time, via $\vec{F}(t,\vec{x})$.

Cor 1 The solution space to the homogeneous system

$$\frac{d\vec{x}}{dt} - A(t)\vec{x} = \vec{0}$$

is n-dim'l

Proof: Let $\vec{z}_i(t)$ be the homogeneous sol'n, with $\vec{z}_i(t_0) = \vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ← i entry

Let $\vec{x}(t)$ be any homogeneous sol'n.

Then $\vec{x}(t_0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ for some vector $\vec{c}$

So $\vec{x}(t)$ must equal $c_1\vec{z}_1(t) + \dots + c_n\vec{z}_n(t)$

since both fcns are sol'ns to the same IVP for the homogeneous DE, so uniqueness thm says they're the same fcn.

Thus $\{\vec{z}_1, \dots \vec{z}_n\}$ span sol'n space.

They're independent because

$$c_1\vec{z}_1(t) + \dots + c_n\vec{z}_n(t) \equiv 0$$

$$\Rightarrow c_1\vec{z}_1(t_0) + \dots + c_n\vec{z}_n(t_0) = 0$$

$$\Rightarrow \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Def If $\vec{y}_1(t), \dots \vec{y}_n(t)$ are vector-valued functions with codomain $\mathbb{R}^n$

then $\begin{bmatrix} \vec{y}_1 \,\big|\, \vec{y}_2 \,\big|\, \cdots \,\big|\, \vec{y}_n \end{bmatrix}$ is the Wronskian matrix

and its det, called $W(\vec{y}_1, \dots \vec{y}_n)$ is the Wronskian,

Theorem: If $\{\vec{y}_1, \dots \vec{y}_n\}$ each solve $\frac{d\vec{x}}{dt} - A(t)\vec{x} = \vec{0}$ on I

then either $W(y_1, \dots y_n) \equiv 0$ on I

or $W(y_1, \dots y_n) \neq 0 \ \forall t \in I$.

The first case ~~is~~ means $\{y_1, \dots y_n\}$ are dependent, the second case means they're independent.

proof: (Use $\exists!$, as above)

<u>Cor 2</u> (the real reason ∃! theorem holds for n^{th} order linear DE IVP.)

Let $L(y) := y^{(n)} + a_{n-1}(t)\, y^{(n-1)} + \cdots + a_1(t)\, y' + a_0(t)\, y$, $a_j \in C(I)$

$f \in C(I)$

Let $t_0 \in I$, $\vec{b} = [b_0, b_1, \ldots b_{n-1}] \in \mathbb{R}^n$

Then ∃! soln $y(t)$ to IVP: , $y \in C^n(I)$.

$$\text{IVP1} \begin{cases} L(y) = f & \text{in } I \\ y(t_0) = b_0 \\ y'(t_0) = b_1 \\ \vdots \\ y^{(n-1)}(t_0) = b_{n-1} \end{cases}$$

<u>proof</u>: This IVP is equivalent to the first order system IVP for $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$:

$$\text{IVP2} \begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_n' = f - a_{n-1} x_{n-1} - a_{n-2} x_{n-2} - \cdots - a_0 x_1 \\ x_1(t_0) = b_0 \\ x_2(t_0) = b_1 \\ \vdots \\ x_n(t) = b_{n-1} \end{cases}$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ & & & \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{bmatrix}$$

i.e. ∃! y sol'n to IVP 1

iff

$y = x_1$
$y' = x_2$
$\vdots$
$y^{(n-1)} = x_n$ is ! soln to IVP2

Thus <u>Cor 2</u>, the Chapter 3 ∃! theorem,
follows from the linear systems ∃! theorem in Chapter 4 (page 1 theorem).