Math 2280-1
Wed October 1

63.5 (not on Fri. exam.)
We are now experts at finding the general homogeneous soln.
to $n^{th}$ order const. coeff. DE's, i.e.

$$L = y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y$$

with $a_j = \text{const.}, j = 0, 1, \ldots, n-1$

We can always construct a basis for
the $n$-dim'l space of solns to

$$L(y) = 0.$$

63.5 contains algorithms for finding particular solns for the non-homogeneous
DE's:

$$L(y_p) = f$$

(and then the full soln will be $y = y_p + y_H$)

Method 1: "Method of undetermined coeff's" (i.e. guessing)
(Actually depends on linear algebra)

**Example:**

$y' + 4y = e^x$

$y_H = e^{-4x}$

Try $y_p = C e^x$ since $L(e^x) = p(x)e^x$

so for $V = \text{span } \{e^x\}$,

$L: V \to V$.

4. $y_p = C e^x$

1. $y_p = C e^x$

$L(y_p) = 5 C e^x$  \[ \text{want } e^x \]

$C = \frac{1}{5}$

$y = y_p + y_H = \frac{1}{5} e^x + C e^{-4x}$
example 2: \( y' + 4y = \cos 2x \)

for \( V = \text{span} \{ \cos 2x, \sin 2x \} \)

\( L: V \to V \)
on \( V \), \( \ker L = \{ 0 \} \)
(since on \( C'(\mathbb{R}) \), \( \ker L = \text{span} \{ e^{-4x} \} \))

So \( \exists! v \in V \) s.t. \( Lv = \cos 2x \)
(rank + nullity theorem).

clever 2270 graduate's way. (equivalent, but quicker.)

<table>
<thead>
<tr>
<th>basis</th>
<th>( { \cos 2x, \sin 2x } )</th>
</tr>
</thead>
</table>

matrix for \( L \) wrt \( \beta \)

\[
[L]_\beta = \begin{bmatrix}
[L(\cos 2x)]_{\beta} & [L(\sin 2x)]_{\beta}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
4 & 2 \\
-2 & 4
\end{bmatrix}
\]

so we want to solve this system for the columns!

Ans: \( y_p = \frac{1}{3} \cos 2x + \frac{1}{10} \sin 2x \)

example 3: find \( y_p \) for \( L(y_p) = x + 2 \)

hint: \( \text{let } V = P_1 = \text{span} \{ 1, x \} \).

Ans: \( y_p = \frac{1}{4} x + \frac{7}{10} \)
Example 4. Using previous 3 examples, find the full sol'n to

\[ y' + 4y = 2e^x - 3\cos x - 5x - 10 \]

hint: superposition.

See guessing table page 202

<table>
<thead>
<tr>
<th>( f(x) ) (in a piece of ( f ))</th>
<th>guess</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^{rx} )</td>
<td>( C e^{rx} )</td>
</tr>
<tr>
<td>( e^{rx} ) (Acos bx + Bsin bx)</td>
<td>( e^{rx} (Acos bx + Bsin bx) )</td>
</tr>
<tr>
<td>( P_n(x) )</td>
<td>( Q_n(x) )</td>
</tr>
<tr>
<td>( P_n(x) e^{rx} )</td>
<td>( Q_n(x) e^{rx} )</td>
</tr>
<tr>
<td>etc</td>
<td>etc</td>
</tr>
</tbody>
</table>

Fix it recipe if \( L \) has non-trivial kernel for original \( V \):

\[
\text{If } \quad \forall x \neq a, \quad f(x) = 0 \Rightarrow |(a, f(a))| = 0
\]

at least one term in guess solves homogeneous eq'n, multiply entire guess by \( x^5 \) where \( 5 \) is the smallest counting \# st. no term in new guess solves homogeneous DE.
Example

\[ L(y) = y'' + 2y' + y \]

solve

\[ L(y) = e^x \]

notice \[ p(r) = (r^2 + 2r + 1) = (r+1)^2 \]

\[ \text{ans: } y_p = \frac{1}{2} xe^x \]
Review sheet for exam 1

1st order ODE's. Chapters 1.1-1.5  
2.1-2.3  (not 1.6, 2.4-2.6)

- Recognize and solve DE's and IVP's for
  \[
  \frac{dy}{dx} = f(x), \quad y(0) = \text{initial condition} \]
  
  \[
  \frac{dy}{dx} = f(x), \quad f(0) = \text{initial condition} \]
  
  \[
  \frac{dy}{dx} = f(y), \quad g(0) = \text{initial condition} \]
  
- slope fields, integral for IVP
  
- phase portraits
  
- applications & modeling
  
  - separable
  
  - exp growth & decay
  
  - Newton's law of cooling
  
  - Torricelli
  
  - logistic/doomsday ext./harvesting (9.1.1)
  
  - acceleration with drag terms (9.2.1)
  
- linear
  - mainly mixing problems (9.1.9)
  - also some of 9.2.3 models
  - if \( F(x) \) is const, i.e.,
    \[
    y' + ay = f
    \]
    
    this theory connects to chapter 3
    \[
    y = y_1 + y_2 \]
    
- Higher order linear DE's (9.1-9.4)
  
  \[
  \begin{aligned}
  L(y) := & y^{(m)} + a_1 y^{(m-1)} + \cdots + a_n y' + a_{n+1} y \\
  & \text{is called linear}
  \end{aligned}
  \]
  
  \[
  \text{why the general soln to } L(y) = f \text{ is } y = y_1 + y_2 \text{? general superposition}
  \]
  
- Dimension of soln space to \( L(y) = 0 \)
  
- (and its relationship to \( \exists! \) thm for IVP)
  
- linear ind. & dep. of funs
  
- Wronskian matrix & determinant
  
- Solving \( L(y) = 0 \) if \( L \) has constant coeffs \( a_i \):
  
  \[
  \begin{aligned}
  & \text{if } p(x) \text{, real roots (distinct, repeated), complex roots (distinct & repeated)} \\
  & \text{uses complex exponentials, Euler, cos & sin addition angle formulas}
  \end{aligned}
  \]
  
- Mechanical vibrations
  
  - pendulum & spring models
  
  - simple harmonic motion
  
  - amp, phase, amp. freq, freq, period, etc. ABC triangle
  
  - 3 kinds of damping & corresponding solution types.