Math 2280-1
Wed October 1

practice exams & sol'ns
are posted!

§3.5 (not on Fri. exam.)

We are now experts at finding the general homogeneous sol'n to n^{th} order const. coeff DE's, i.e.

for

$$L = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y \qquad \text{with } a_j = \text{const}, j = 0, 1, \ldots n-1$$

We can always construct a basis for the n-dim'l space of sol'ns to

$$L(y) = 0.$$

§3.5 contains algorithms for finding particular sol'ns for the non-homogeneous DE,

$$L(y_P) = f$$

(and then the full sol'n will be $y = y_P + y_H$)

Method 1 "Method of undetermined coeffs" (i.e. guessing)

(Actually depends on linear algebra)

example 1 : $y' + 4y = e^x$

$y_H = ce^{-4x}$

try $y_P = Ce^x$ since $L(e^{rx}) = p(r)e^{rx}$

so for $V = \text{span}\{e^x\}$,

$L: V \to V$.

$$\begin{aligned} 4(\quad & y_P = Ce^x \\ 1(\quad & y_P' = Ce^x \\ \hline & L(y_P) = 5Ce^x \overset{\text{want}}{=} e^x \end{aligned}$$

$C = 1/5$

$$y = y_P + y_H = \tfrac{1}{5}e^x + Ce^{-4x}$$

example 2 $y' + 4y = \cos 2x$

need to include this fcn!

for $V = \text{span}\{\cos 2x, \sin 2x\}$

$L : V \to V$

on $\underline{V}$, $\ker L = \{0\}$ (since on $C^1(\mathbb{R})$, $\ker L = \text{span}\{e^{-4x}\}$)

so $\exists!\, v \in V$ s.t. $Lv = \cos 2x$ (rank + nullity thm!).

book's way.

$$4(\ \text{try } y_p = A\cos 2x + B\sin 2x$$
$$1(\ \ y_p' = -2A\sin 2x + 2B\cos 2x$$

$$L(y_p) = \cos 2x\,[4A+2B] + \sin 2x\,[-2A+4B] \overset{\text{want}}{=} \cos 2x\,[1] + \sin 2x\,[0]$$

clever 2270 graduate's way. (equivalent, but quicker!)

basis $\mathcal{B} = \{\overset{f_1}{\cos 2x}, \overset{f_2}{\sin 2x}\}$

matrix for L wrt $\mathcal{B}$ $\longrightarrow$ $[L]_{\mathcal{B}} = \left[\, [L(f_1)]_{\mathcal{B}} \,\middle|\, [Lf_2]_{\mathcal{B}} \,\right]$

$$= \begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix}$$

$\uparrow$ $L(\cos 2x) = 4\cos 2x - 2\sin 2x$

$\uparrow$ $[L(\sin 2x)]_{\mathcal{B}}$

so we want to solve this eqtn for the coords of the sol'n!

$$\begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 4 & -2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 1/10 \end{bmatrix}$$

$$\boxed{y_p = \frac{1}{5}\cos 2x + \frac{1}{10}\sin 2x}$$

example 3: find y_p for $L(y_p) = x + 2$

hint: let $V = P_1 = \text{span}\{1, x\}$.

ans: $y_p = \frac{1}{4}x + \frac{7}{16}$

Example ④ Using previous 3 examples, find the <u>full</u> sol'n to

$$y' + 4y = 2e^{x} - 3\cos x - 5x - 10$$

hint: superposition.

See guessing table page 202

f(x) (or a piece of f)	guess	
e^{rx}	Ce^{rx}	$V = \text{span}\{e^{rx}\}$ $L: V \to V$ L is 1-1 & onto iff r is <u>not</u> a root of charact poly.
$e^{ax}(A\cos bx + B\sin bx)$	$e^{ax}(D\cos bx + E\sin bx)$	$V = \text{span}\{e^{ax}\cos bx, e^{ax}\sin bx\}$ L is 1-1 & onto iff $a \pm bi$ <u>not</u> roots of char. poly
$P_n(x)$	$Q_n(x)$	$V = \mathbb{P}_n = \cancel{\{}\text{span}\{1, x, \ldots x^n\}$ L is 1-1 & onto iff 0 is <u>not</u> a root of c.p.
$P_m(x)e^{rx}$	$Q_m(x)e^{rx}$	$V = \text{span}\{e^{rx}, xe^{rx}, \ldots x^m e^{rx}\}$ works if r is <u>not</u> a root of charact poly.
etc		

Fixit recipe if L has non-trivial kernel for original V:

If

at least one term in guess solves homogeneous eqtn, multiply entire guess by x^s where s is the smallest counting # s.t. no term in new guess solves homogeneous DE !!

Example

$$L(y) = y'' + 2y' + y$$

solve

$$L(y) = e^{-x}$$

notice $p(r) = (r^2 + 2r + 1) = (r+1)^2$

ans: $y_p = \frac{1}{2}x^2e^{-x}$!

Review sheet for exam 1.

1^{st} order ODE's. Chapters 1.1-1.5, 2.1-2.3 (not 1.6, 2.4-2.6)

- Recognize and solve DE's and IVP's for
  - $\frac{dy}{dx} = f(x)$ §1.2 antidiff
  - $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ §1.4 separable
  - $\frac{dy}{dx} + P(x)y = Q(x)$ §1.5 linear
- slope fields, ∃! for IVP §1.3, 2.2
  phase portraits.
  - geometric meaning of $\frac{dy}{dx} = f(x,y)$ for the graph $y = y(x)$
  - sketch slope fields using isoclines
  - ∃! thm for IVP
  - autonomous DE's $\frac{dy}{dx} = f(y)$
    - equilibrium sol'ns
    - phase portraits
    - stability
- applications & modeling
  - separable
    - exp growth & decay
    - Newton's law of cooling
    - Toricelli
    - logistic / doomsday ext/harvesting (§2.1)
    - acceleration with drag terms (§2.3)
  - linear
    - mainly mixing problems (§1.5)
    - also some of §2.3 models
    - if $P(x)$ is const, i.e.
      $y' + a_0 y = f$
      this theory connects to chapter 3 $y = y_p + y_H \cdots$
      use of e^{rt}, etc.

Higher order linear DE's (3.1-3.4)

- Why $L(y) := y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_1(x) y' + a_0(x) y$
  is called linear
  Why the general sol'n to $L(y) = f$ is $y = y_p + y_H$; more general superposition
  Dimension of sol'n space to $L(y) = 0$
  (and its relationship to ∃! thm for IVP)
  linear ind & dep of fcns
  - Wronskian matrix & determinant
- Solving $L(y) = 0$ if L has constant coeffs a_j
  - e^{rx}, $p(r)$, real roots (distinct, repeated), complex roots (distinct & repeated)
  - using complex exponentials, Euler, cos & sin addition angle formulas
- Mechanical vibrations
  - pendulum & spring models
  - simple harmonic motion (amp, phase, ang. freq, freq, period, etc. ABC triangle)
  - the three kinds of damping and corresponding solution types.