Math 2280-1
Friday Nov. 7

$\boxed{\text{2nd exam is next Friday, i.e. Nov 14, a week from today. It will cover thru Chapter 5, including 5.6. So make the postponed 5.6 HW due Wednesday next week, Nov. 12, and there will be no homework due Monday Nov. 17}}$

- look over notes from the extended example Wed. (see below) ... this should just take a couple minutes.. page 3 is new
- discuss (easier?) way to find FMS $\Phi(t)$ when eigenvalues are defective, i.e. without constructing chains.
- Go over § 5.6 "undetermined coeff's (Wed notes), and variation of parameters (today).

(Next week after we've finished 5.6, we'll prove the dimension theorems for generalized eigenspaces, discuss the context of chains and "Jordan form" for matrices. This material will be mathematical enrichment, not on exam.)

Wed.
Example recap
(for browsing)

$$N=\begin{bmatrix}0&1&2\\0&0&-1\\0&0&0\end{bmatrix}\qquad N^2=\begin{bmatrix}0&0&-1\\0&0&0\\0&0&0\end{bmatrix}\qquad N^3=\left[\,O\,\right]$$

so N is nilpotent ($N^k=0$ some k)

thus
$$e^{Nt}=I+Nt+N^2\frac{t^2}{2!}+[O]$$
$$=\begin{bmatrix}1&t&2t+t^2/2!\\0&1&-t\\0&0&1\end{bmatrix}$$

If $A=\lambda I+N$ (N nilpotent)

then
$$e^{At}=e^{(\lambda I+N)t}=e^{\lambda It}e^{Nt}$$
because λI and N commute
$$=e^{\lambda t}I\,e^{Nt}$$
since exponentials of diagonal matrices are diagonal matrices of exponentials
$$=e^{\lambda t}e^{Nt}$$

So if $A=\begin{bmatrix}3&1&2\\0&3&-1\\0&0&3\end{bmatrix}=3I+N$ (as above)

$$\boxed{e^{At}=e^{3t}e^{Nt}=e^{3t}\begin{bmatrix}1&t&2t+t^2/2!\\0&1&-t\\0&0&1\end{bmatrix}}$$

Chains approach, same A

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & & \\ 0 & 3-\lambda & \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^3 \, ;$$

$\lambda = 3$ is only eigenvalue.
alg. mult is 1, since

$$[A-3I \,\vdots\, 0] = [N \,\vdots\, 0] = \left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\downarrow$ rref

$$\begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} v_3 = 0 \\ v_2 = 0 \\ v_1 = s \end{matrix} \qquad \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

E_3 is only one-dim'l.

But by (to be proven next week) theorem,

$G_3 := \ker(A-3I)^3 \leftarrow$ alg. mult here

has $\dim(G_3) =$ alg mult $= 3$.

Thus $G_3 = \mathbb{R}^3$.

We search for a chain

$$\begin{matrix} A-3I \\ \begin{matrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{matrix} \end{matrix} \qquad \begin{matrix} (A-3I)^2 \\ \begin{matrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \end{matrix} \qquad \begin{matrix} (A-3I)^3 \\ 0 \end{matrix}$$

$\vec{v}_3 = \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ sats $(A-3I)^2 \vec{v}_3 \neq \vec{0}$

$\vec{v}_2 = (A-3I)\vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

$\vec{v}_1 = (A-3I)\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$; $\left((A-3I)\vec{v}_1 = \vec{0}\right)$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ chain of length 3.

yields FMS

$$\Phi(t) = \left[e^{3t}\vec{v}_1 \;\middle|\; e^{3t}(t\vec{v}_1 + \vec{v}_2) \;\middle|\; e^{3t}\left(\tfrac{t^2}{2}\vec{v}_1 + t\vec{v}_2 + \vec{v}_3\right) \right] = e^{3t}\begin{bmatrix} -1 & -t+2 & -\frac{t^2}{2}+2t \\ 0 & -1 & -t \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Phi(0) = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Phi(0)^{-1} = \begin{bmatrix} -1 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{At} = \Phi(t)\Phi(0)^{-1}$$

$$= e^{3t}\begin{bmatrix} -1 & -t+2 & -\frac{t^2}{2}+2t \\ 0 & -1 & -t \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= e^{3t}\begin{bmatrix} 1 & t & -t^2/2+2t \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}$$ agrees with page 1

Chain-free way to get $\Phi(t)$: $|A-\lambda I|$ has factor $(\lambda-\lambda_j)^{k_j}$ with k_j = alg mult

Then $G_{\lambda_j} = \ker(A-\lambda_j)^{k_j}$ has dim k_j

pick a basis $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_{k_j}$ for this generalized eigenspace.

Then $\vec{x}_\ell(t) = e^{At}\vec{v}_\ell$ is the soltn to $\begin{cases} \vec{x}'(t) = A\vec{x} \\ \vec{x}(0) = \vec{v}_\ell \end{cases}$

$$= e^{\lambda_j I t} e^{(A-\lambda_j I)t}\vec{v}_\ell$$

$$= e^{\lambda_j t}\left[e^{(A-\lambda_j I)t}\vec{v}_\ell\right]$$

$$= e^{\lambda_j t}\left[I + (A-\lambda_j I)t + \ldots\right]\vec{v}_\ell$$

$$= e^{\lambda_j t}\left[\vec{v}_\ell + t(A-\lambda_j I)\vec{v}_\ell + \frac{t^2}{2}(A-\lambda_j I)^2\vec{v}_\ell + \ldots \frac{t^{k-1}}{(k-1)!}(A-\lambda_j I)^{k-1}\vec{v}_\ell + \vec{0}\right]$$

finite sum! $\qquad \vec{v}_\ell \in \ker(A-\lambda_j I)^k$!

this gives k_j soltns "for" λ_j
amalgamate to get $\Phi(t)$!

example con't $\quad A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ (still) $\quad G_3$ has dim 3, i.e. is $\mathbb{R}^3$
so, pick $\vec{v}_1 = \vec{e}_1$, $\vec{v}_2 = \vec{e}_2$, $\vec{v}_3 = \vec{e}_3$.

$$\vec{x}_1(t) = e^{At}\vec{e}_1 = e^{3t}e^{(A-3I)t}\vec{e}_1 = e^{3t}\Big[I\vec{e}_1 + \underbrace{(A-3I)t\vec{e}_1}_{\text{already zero}} + \ldots\Big] = e^{3t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\vec{x}_2(t) = e^{At}\vec{e}_2 = e^{3t}e^{(A-3I)t}\vec{e}_2 = e^{3t}\Big[I\vec{e}_2 + \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}t\vec{e}_2 + \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\frac{t^2}{2}\vec{e}_2}_{\text{already zero}} + \ldots\Big]$$

$$= e^{3t}\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{x}_3(t) = e^{At}\vec{e}_3 = e^{3t}e^{(A-3I)t}\vec{e}_3$$

$$= e^{3t}\Big[I\vec{e}_3 + \underbrace{\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}t\vec{e}_3}_{\begin{bmatrix} 2t \\ -t \\ 0 \end{bmatrix}} + \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\frac{t^2}{2}\vec{e}_3}_{\begin{bmatrix} -t^2/2 \\ 0 \\ 0 \end{bmatrix}} + \vec{0}\Big]$$

$$= e^{3t}\begin{bmatrix} 2t - t^2/2 \\ -t \\ 1 \end{bmatrix}$$

$$\Phi(t) = \left[\vec{x}_1 \middle| \vec{x}_2 \middle| \vec{x}_3\right] = e^{3t}\begin{bmatrix} 1 & t & 2t - t^2/2 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}$$; also equals e^{At}, in this case

Variation of parameters to find $\vec{x}_p(t)$ for the system

(this includes the old vari. p. for n^{th} order linear DE's!!)

$$\vec{x}'(t) - P(t)\vec{x} = \vec{f}(t),$$

given FMS $\Phi(t)$ for homog. sys. $\vec{x}'(t) = P(t)\vec{x}$.

i.e. $\Phi'(t) = P(t)\Phi$

and Φ^{-1} exists $(\forall t)$

Try $\vec{x}_p(t) = \Phi(t)\vec{u}(t)$

$$\vec{x}_p'(t) = \Phi'\vec{u} + \underbrace{\Phi\vec{u}'}_{\text{set this } = \vec{f}(t)}$$

i.e. $\boxed{\vec{u}' = \Phi^{-1}\vec{f}}$

(antidifferentiate to get a $\vec{u}$)

this equals $(P(t)\Phi)\vec{u}$
$= P(t)(\Phi\vec{u})$
$= P(t)\vec{x}$

thus, if $\vec{u}' = \Phi^{-1}(t)\vec{f}(t)$
then $\vec{x}_p(t) = \Phi(t)\vec{u}(t)$ is a particular sol'tn!

Thus, (continuing), the full sol'tn is

$$\vec{x}(t) = \Phi(t)\left(\int \Phi^{-1}(t)\vec{f}(t)\right)$$

notice the arbitrary const. of integration $\vec{c}$, gives the general homogeneous sol'tn $\Phi(t)\vec{c}$.

Notice, if our system came from an n^{th} order linear DE, this is the formula we came up with! We wrote it

$$\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = W^{-1}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ f \end{bmatrix}$$

If you want to solve

$$\begin{cases} \vec{x}'(t) = P(t)\vec{x} + \vec{f}(t) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

we may write the sol'tn as

$$\boxed{\vec{x}(t) = \Phi(t)\Phi(t_0)^{-1}\vec{x}_0 + \Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)\vec{f}(s)\,ds}$$

If $\Phi(t) = e^{At}$, for $\vec{x}'(t) = A\vec{x} + \vec{f}$ (i.e. $P(t) = A$, const)
and $t_0 = 0$
sol'n above is

$$\boxed{\vec{x}(t) = e^{At}\vec{x}_0 + e^{At}\int_0^t e^{-As}\vec{f}(s)\,ds}$$

(can also derive this as in chptr 1, !!)

$$\vec{x}' - A\vec{x} = \vec{f}$$
$$(e^{-At}\vec{x})' = e^{-At}\vec{f}$$
$$\int_0^t$$
$$e^{-At}\vec{x}(t) - \vec{x}_0 = \int_0^t e^{-As}\vec{f}(s)$$

examples in text (and Monday)