Math 2280-1
Wednesday Nov. 5

e^{At}: ① unique sol'n to $\begin{cases} X' = AX \\ X(0) = I \end{cases}$

② $\Phi(t)\Phi^{-1}(0)$ where $\Phi(t)$ is any FMS for $\frac{d\vec{x}}{dt} = A\vec{x}$

③ $I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots + \frac{t^k}{k!}A^k + \cdots$

<u>if A is diagonalizable</u> (i.e. $\exists$ $\mathbb{R}^n$ eigenbasis $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$)
$A\vec{v}_j = \lambda_j \vec{v}_j$

then
$$\Phi(t) = \left[e^{\lambda_1 t}\vec{v}_1 \,\middle|\, e^{\lambda_2 t}\vec{v}_2 \,\middle|\, \cdots \,\middle|\, e^{\lambda_n t}\vec{v}_n \right] = \underbrace{\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \vec{v}_n \right]}_{S} \underbrace{\begin{bmatrix} e^{\lambda_1 t} & & O \\ & e^{\lambda_2 t} & \\ O & & \ddots \ e^{\lambda_n t} \end{bmatrix}}_{e^{\Lambda t}}$$

$$e^{At} = S e^{\Lambda t} S^{-1} \qquad \left(= \Phi(t)\Phi(0)^{-1}\right).$$

which we also get from the Taylor series def, since
$$\Lambda = S^{-1}AS$$
so $S\Lambda S^{-1} = A$
$$\text{so } e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \cdots$$
$$= S\left[I + t\Lambda + \frac{t^2}{2!}\Lambda^2 + \frac{t^3}{3!}\Lambda^3 + \cdots\right]S^{-1}$$
$$= S e^{t\Lambda} S^{-1} \ \checkmark$$

<u>if A is not diagonalizable</u> (defective eigenspaces)
then you can construct a $\Phi(t)$ using chains, and compute $\Phi(t)\Phi(0)^{-1}$,
or you can use power series tricks
– e.g. <u>do example 4</u> on page 2 yesterday.

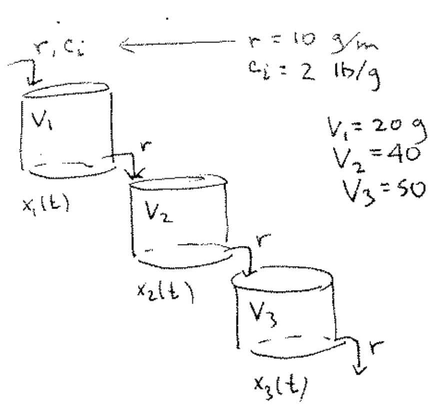
(I hope to write some more complete notes for this case (Friday?), and to explain how it relates to "Jordan canonical form", as well as provide proofs for the chain decomposition algorithms.)

§5.6 Nonhomogeneous linear systems of DE's
(will use e^{At} & FMS's)

$$\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{f}(t)$$

- undetermined coeff's in case $P(t) = A$ (const matrix) and $\vec{f}(t)$ is one of good guessing functions
- variation of parameters (☺), for general $P(t)$, $\vec{f}(t)$, provided you already have a FMS $\Phi(t)$ for the homogeneous system $\frac{d\vec{x}}{dt} = P(t)\vec{x}$.

Example p. 363



$\vec{x}(0) = \vec{0}$.

Explain why the 1st order system for $\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ is

$$* \qquad \frac{d\vec{x}}{dt} = \begin{bmatrix} -.5 & 0 & 0 \\ .5 & -.25 & 0 \\ 0 & .25 & -.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix}$$

general sol'n is

$$\vec{x}(t) = \vec{x}_H(t) + \vec{x}_P(t)$$

as usual.

eigenvects(A) = [−.5, 1, {[5, −6, 5]}], [−.25, 1, {[0, 1, −5]}], [−.2, 1, {[0, 0, 1]}] (Maple)

So, for $x_H(t)$ a FMS $\Phi(t)$ is

$$\Phi(t) = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

and general sol'n to $\vec{x}_H(t) = \Phi(t)\vec{c}$.

For $\vec{x}_p(t)$ we guess a constant vector, since

$$L(\vec{x}) := \vec{x}' - A\vec{x}$$

transforms constant vectors to constant vectors [This is undetermined coeff's easy case for systems!]

If $\vec{x}_p(t) = \vec{k}$, then plug into DE: $\vec{x}' = A\vec{x} + \vec{b}$, page 1

$$\vec{0} = \begin{bmatrix} -.5 & 0 & 0 \\ .5 & -.25 & 0 \\ 0 & .25 & -.2 \end{bmatrix} \vec{k} + \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix} ;$$

$$\vec{k} = A^{-1} \begin{bmatrix} -20 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 40 \\ 80 \\ 100 \end{bmatrix} \quad \text{(Maple)}$$

general sol'n

$$\vec{x}(t) = \vec{k} + \Phi(t)\vec{c} \qquad (\vec{x}_p + \vec{x}_H), \quad \text{with } \Phi(t) = \left[e^{-.5t}\begin{bmatrix} 5 \\ -6 \\ 5 \end{bmatrix}, e^{-.25t}\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}, e^{-.2t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]$$

If $\vec{x}(0) = \vec{0}$ (as in example),

$$\vec{0} = \vec{k} + \Phi(0)\vec{c}$$

$$\vec{c} = \Phi(0)^{-1}(-\vec{k}) = \begin{bmatrix} -40/3 \\ -160 \\ -\frac{2500}{3} \end{bmatrix} \quad \text{(Maple)}$$

so $$\vec{x}(t) = \begin{bmatrix} 40 \\ 80 \\ 100 \end{bmatrix} - \frac{40}{3} e^{-.5t} \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} - 160 e^{-.25t} \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} - \frac{2500}{3} e^{-.2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

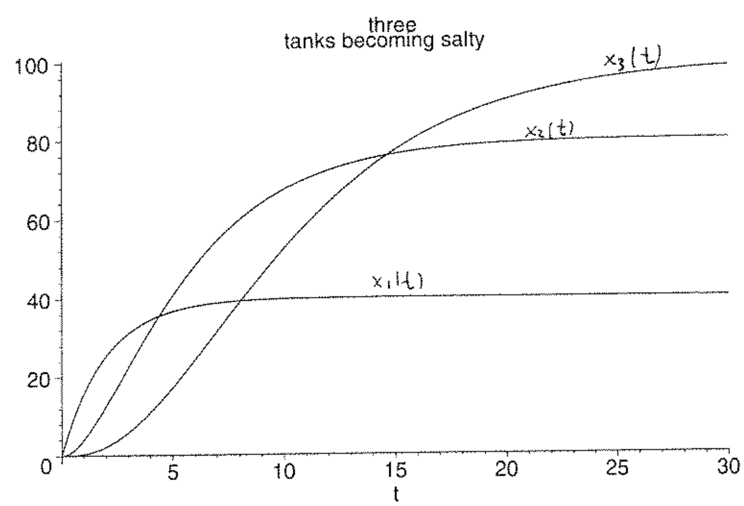
```
> x1:=t->40-40*exp(-.5*t);
  x2:=t->80+80*exp(-.5*t)-160*exp(-.25*t);
  x3:=t->100-200/3*exp(-.5*t)+5*160*exp(-.25*t)-
     2500/3*exp(-.2*t);
```

$$x1 := t \to 40 - 40\,e^{(-0.5\,t)}$$

$$x2 := t \to 80 + 80\,e^{(-0.5\,t)} - 160\,e^{(-0.25\,t)}$$

$$x3 := t \to 100 - \frac{200}{3}\,e^{(-0.5\,t)} + 800\,e^{(-0.25\,t)} - \frac{2500}{3}\,e^{(-0.2\,t)}$$

```
> plot({x1(t),x2(t),x3(t)},t=0..30,color=black,title=`three
  tanks becoming salty`);
```



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but, for harder nonhomogeneous problems, you'll prefer to use variation of parameters! ☺