Math 2280-1
Tuesday Nov. 4

Recall,
for the 1st order system

$*\quad \vec{x}' = A\vec{x} \qquad A_{n\times n}$ const matrix

- $\Phi(t)$ is a FMS (or FSM if I'm dyslexic)

  means
  $$\begin{cases} \Phi' = A\Phi \\ \Phi(t) \text{ invertible (iff } \Phi(0) \text{ invertible} \end{cases}$$
  equivalent to the columns of $\Phi(t)$ are a basis for sol'ns to $*$

- If $\Phi(t)$ is a FMS, so is $\Phi(t)C$ whenever C^{-1} exists.

- Sol'n to IVP
  $$\begin{cases} \vec{x}' = A\vec{x} \\ \vec{x}(0) = \vec{x}_0 \end{cases} \quad \text{is} \quad \vec{x}(t) = \Phi(t)\left(\Phi^{-1}(0)\,\vec{x}_0\right)$$

- $e^{At} :=$ the FMS solving
  $$\begin{cases} X' = AX \\ X(0) = I \end{cases}$$
  (after all, for scalars, e^{at} solves $\begin{cases} x'(t) = ax \\ x(0) = 1 \end{cases}$)

- Thus $e^{At} = \Phi(t)\Phi^{-1}(0)$ for any FMS $\Phi(t)$

then finish pages 4, 5 Monday!
then try examples on page 2.

Sometimes the power series is easiest:

example 2 $\quad A = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$; $\Lambda^2 = \begin{bmatrix} \lambda_1^2 & & 0 \\ & \lambda_2^2 & \\ & \ddots & \\ 0 & & \lambda_n^2 \end{bmatrix}$; $\Lambda^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}$

so $e^{At} = I + t\Lambda + \frac{t^2}{2!}\Lambda^2 + \cdots$

$$= \begin{bmatrix} 1 + t\lambda_1 + \frac{t^2}{2!}\lambda_1^2 + \cdots & & & \\ & 1 + t\lambda_2 + \frac{t^2}{2!}\lambda_2^2 + \cdots & & \\ & & \ddots & \\ 0 & & & 1 + t\lambda_n + \cdots \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

example 3

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

power series way

$A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$

could you do this with a FMS?

$\Rightarrow A^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A^4 = I$

$$e^{At} = I + tA + \frac{t^2}{2!}(-I) + \frac{t^3}{3!}(-A) + \frac{t^4}{4!}I + \frac{t^5}{5!}A + \cdots$$

$$= \cos t \, I + \sin t \, A$$

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \quad !!$$

example 4 $\quad [N]$ is called nil potent if $N^k = 0$ for some k.

then $e^{Nt} = I + tN + \cdots + \frac{t^{k-1}}{(k-1)!}N^{k-1} + 0 \quad !$

example $\quad N = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad N^2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad N^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$e^{Nt} = I + tN + \frac{t^2}{2}N^2 = \begin{bmatrix} 1 & t & 2t - t^2/2 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}$$

If $A = \lambda I + N$ where N is nilpotent, then

$$e^{At} = e^{(\lambda I + N)t}$$

$$= \underbrace{e^{\lambda I t}}_{e^{\lambda t} I \text{ (example 2)}} e^{Nt} \qquad \left[e^{A+B} = e^A e^B \text{ whenever } A \text{ and } B \text{ commute; see HW} \right]$$

e.g. $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow e^{At} = e^{3t}\begin{bmatrix} 1 & t & 2t - t^2/2 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix} \quad !$

Universal product rule

Recall the 1-var. proof of the product rule, write $*$ for "times", and see the generality in which it really applies: (t is a real variable)

$$D_t(f*g)(t) = \lim_{\Delta t \to 0} \left[f(t+\Delta t) * g(t+\Delta t) - f(t)*g(t) \right] \frac{1}{\Delta t}$$

Def. of deriv

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[[f(t+\Delta t) - f(t)] * g(t+\Delta t) + f(t) * [g(t+\Delta t) - g(t)] \right]$$

mult $*$ distributes over addition on both sides

$$= \lim_{\Delta t \to 0} \left[\frac{1}{\Delta t}(f(t+\Delta t) - f(t)) \right] * g(t+\Delta t)$$
$$+ \lim_{\Delta t \to 0} f(t) * \left[\frac{1}{\Delta t}(g(t+\Delta t) - g(t)) \right]$$

scalar mult distributes into either factor of the product

$$\boxed{D_t(f*g)(t) = f'(t)*g(t) + f(t)*g'(t)}$$

limit of product is product of limits. & differentiable fcns are continuous

examples
- function times fcn (1210)
- dot product } 2210
- cross product } 2210
- scalar fcn times vector fcn } 2280
- matrix fcn times vector fcn } 2280
- matrix fcn times matrix fcn } 2280

not an example composition, $f \circ g$!
(which is why we have the chain rule.)