Math 2280-1
Monday Nov. 3

HW for Nov 10
5.4 1, 7, 11, 29
5.5 1, 3, 11, 23, 33, 36
5.6 1, 13, 15, 19, 23
See our homework page for more specific directions re technology

§5.5 Matrix exponentials & linear systems of DE's.

(first finish the example from Wednesday notes).

Def: Consider

$*$ $\vec{x}' = A\vec{x}$

$A_{n\times n}$ constant matrix

If $\{\vec{x}_1(t), \vec{x}_2(t), \ldots \vec{x}_n(t)\}$ is a basis of sol'ns to $*$

then

$$\Phi(t) = \left[\ \vec{x}_1(t)\ \middle|\ \ldots\ \middle|\ \vec{x}_n(t)\ \right]$$ is called a $\left.\begin{array}{l}\text{fundamental}\\ \text{matrix}\\ \text{solution}\end{array}\right\}$ FMS

Notice, this is equivalent to saying $X(t) = \Phi(t)$ solves

$$\begin{cases} X' = A\,X_{n\times n} & \text{(just look column by column!)} \\ X(0) & \text{non-singular (i.e. invertible, rank } n, \det \neq 0) \end{cases}$$

Example 1 p 346

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 10 = (\lambda-5)(\lambda+2)$$

$\lambda = 5$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & -6 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\lambda = -2$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\vec{x}_1(t) = e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \vec{x}_2(t) = e^{-2t}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

possible $$\Phi(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}$$

Theorem If $\Phi(t)$ is a FMS for * then the (unique) sol'n to

$$\text{IVP} \quad \begin{cases} \vec{x}' = A\vec{x} \\ \vec{x}(0) = \vec{x}_0 \end{cases}$$

is $\vec{x}(t) = \Phi(t)\left[\Phi^{-1}(0)\,\vec{x}_0\right]$

pf : the general sol'n to * is

$\vec{x}_H(t) = \Phi(t)\,\vec{c} \quad \left(= \{ c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n \right)$

so the $\vec{x}(t) = \Phi(t)\left[\Phi^{-1}(0)\,\vec{x}_0\right]$ solves *

and, $\vec{x}(0) = \left[\Phi(0)\Phi^{-1}(0)\right]\vec{x}_0 = I\,\vec{x}_0 = \vec{x}_0$

so $\vec{x}(t)$ solves IVP ∎

Example 1 cont'd

solve

$$\begin{cases} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{cases}$$

using page 1 $\Phi(t)$ and theorem above.

ans: $\vec{x}(t) = \frac{3}{7}e^{-2t}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + \frac{2}{7}e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Remark: If $\Phi(t)$ is a FMS and $C$ is invertible,

then so is $\Phi(t)C$ (note - you multiply by $C$ on the right)

since

$$\frac{d}{dt}\left(\Phi(t)C\right) = \Phi'(t)C$$
$$= (A\Phi)C$$
$$= A(\Phi C)$$

and $\Phi(0)C$ is invertible.

$\Phi(t)\Phi^{-1}(0)$ is the best FMS because it is the unique soltn to

$$\begin{cases} \frac{dX}{dt} = AX \\ X(0) = I \end{cases}$$

because $col_j(X)$ is the unique soltn to

$$\begin{cases} \frac{d\vec{x}}{dt} = A\vec{x} \\ \vec{x}(0) = \vec{e}_j \end{cases}$$

It's so special we will call it $e^{At}$!

Notice, the sol'n to the IVP

$$\begin{cases} \frac{d\vec{x}}{dt} = A\vec{x} \\ \vec{x}(0) = \vec{c} \end{cases}$$

is $e^{At}\vec{c}$, "just" like for the Chapter 1 scalar eqtn.

Example 1 cont'd

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \frac{1}{7} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} e^{-2t} + 6e^{5t} & -2e^{-2t} + 2e^{5t} \\ -3e^{-2t} + 3e^{5t} & 6e^{-2t} + e^{5t} \end{bmatrix}$$

But wait!

Didn't you like how we derived Euler's formula?

Here's a different (?) way to define $e^{At}$:

$$A_{n\times n} \qquad e^{A} := I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots + \frac{1}{k!}A^k + \dots$$

(1220: notice this converges for any $A_{n\times n}$, since if ~~the biggest entry~~ $|a_{ij}| \le M \ \forall i,j$

then $|\text{entry}_{ij}(A^2)| \le nM^2$

$|\text{entry}_{ij}(A^3)| \le n^2M^3$

$|\text{entry}_{ij}(A^k)| \le n^{k-1}M^k$

so the $ij$ entry series converges absolutely (dominated by the series for $e^{Mn}$)

$$\boxed{e^{tA} := I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \dots + \frac{t^k}{k!}A^k + \dots}$$

notice, for $X(t) := e^{tA}$

$$\begin{cases} X(0) = I \\ X'(t) = A + \frac{2t}{2!}A^2 + \frac{t^2}{2!}A^3 + \dots + \frac{t^{k-1}}{(k-1)!}A^k + \dots \\ \quad = A\left(I + tA + \frac{t^2}{2!}A^2 + \dots\right) \\ \quad = AX \end{cases}$$

(Maybe in 3220 you'll learn why you can diff this $\infty$ series term by term)

Theorem

Since the power series def of $e^{tA}$ solves $\begin{cases} X' = AX \\ X(0) = I \end{cases}$ (and sol'n is unique)

It must be that

$$\boxed{e^{tA} = \Phi(t)\Phi^{-1}(0) = I + tA + \frac{t^2}{2!}A^2 + \dots + \frac{t^k}{k!}A^k + \dots}$$

Example 1 cont'd (2270!!)

$$A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$S = \left[ \vec{v}_1 \,\middle|\, \vec{v}_2 \right] = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} = S^{-1} A S$$

so $A = S \Lambda S^{-1}$

$A^2 = S\Lambda S^{-1} S \Lambda S^{-1} = S \Lambda^2 S^{-1}$

$A^k = S \Lambda^k S^{-1}$ ⟵ notice $\Lambda^k = \begin{bmatrix} (-2)^k & 0 \\ 0 & 5^k \end{bmatrix}$ is easy to compute!

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^k}{k!}A^k + \cdots$$

$$= I + tS\Lambda S^{-1} + \frac{t^2}{2!} S \Lambda^2 S^{-1} + \cdots + \frac{t^k}{k!} S \Lambda^k S^{-1}$$

$$= S\left[ I + t\Lambda + \frac{t^2}{2!}\Lambda^2 + \cdots + \frac{t^k}{k!}\Lambda^k + \cdots \right] S^{-1}$$

$$= S \begin{bmatrix} \sum_{k=0}^{\infty} \frac{(-2t)^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{(5t)^k}{k!} \end{bmatrix} S^{-1}$$

$$= \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{5t} \end{bmatrix} \frac{1}{7} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & -2e^{-2t} \\ 3e^{5t} & e^{5t} \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} e^{-2t} + 6e^{5t} & -2e^{-2t} + 2e^{5t} \\ -3e^{-2t} + 3e^{5t} & 6e^{-2t} + e^{5t} \end{bmatrix}$$

same as page 3!!!