Math 2280-1

Tuesday Nov. 25

Mostly we'll just discuss Monday's notes (6.4)

If time:

partial HW for Monday Dec. 8

7.1 3, 7, 8, 13, 20, 21, 23, 28

7.2 3, 4, 5, 6, 14, 19, 20, 28, 31

7.3 3, 7, 8, 17, 20, 31

Chapter 7 Laplace transform. (Back to linear DE's ☺)

one of several linear transformations in the field of DE's & PDE's, which have the amazing property of transforming linear DE's into algebraic problems.

for $f : [0,\infty) \to \mathbb{R}$
t

$$\boxed{\underset{\substack{\parallel \\ F(s)}}{\mathcal{L}\{f(t)\}(s)} := \int_0^\infty e^{-st} f(t)\,dt}$$

(it is assumed $|f(t)| \le Ce^{Mt}$ for t large, so that the $F(s)$ integral converges for $s > M$ (or $Re(s) > M$))

Easy fact: $\mathcal{L}$ is linear ($\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\}(s) = c_1 F_1(s) + c_2 F_2(s)$)

Hard fact: $\mathcal{L}$ is 1-1, so has an inverse defined on its image subspace

Easy fact: inverses of linear transformations are always linear too (just take $\mathcal{L}^{-1}$ of both sides of)

Examples

$$\mathcal{L}\{1\}(s) = \int_0^\infty e^{-st}\,1\,dt = \left.\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s} \qquad (Re\, s > 0)$$

$$\mathcal{L}\{e^{at}\}(s) = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{t(a-s)}\,dt = \left.\frac{1}{a-s} e^{t(a-s)}\right]_{t=0}^\infty = \frac{1}{s-a} \qquad (Re\, s > a)$$

$$\cos kt + i \sin kt = e^{ikt} \qquad (k \text{ real})$$

$$\Rightarrow \mathcal{L}\{\cos kt\}(s) + i\,\mathcal{L}\{\sin kt\}(s) = \mathcal{L}\{e^{ikt}\} = \frac{1}{s-ik} \qquad (a = ik)$$

$$= \frac{s+ik}{s^2+k^2}$$

$$\Rightarrow \mathcal{L}\{\cos kt\}(s) = \frac{s}{s^2+k^2}$$

$$\mathcal{L}\{\sin kt\}(s) = \frac{k}{s^2+k^2}$$

Most important:

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty \underbrace{e^{-st}}_{u}\,\underbrace{f'(t)\,dt}_{dv} = \left. e^{-st} f(t)\right]_0^\infty - \int_0^\infty -s e^{-st} f(t)\,dt = -f(0) + sF(s)$$

so, $$\mathcal{L}\{f''(t)\}(s) = -f'(0) + s\,\underbrace{\mathcal{L}\{f'(t)\}(s)}_{-f(0)+sF(s)} = s^2 F(s) - sf(0) - f'(0) \quad \text{etc.}$$

Laplace transform magic:

Example 1

Solve

$$\begin{cases} x''(t) + 4x(t) = 10\cos 3t \\ x(0) = 2 \\ x'(0) = 1 \end{cases}$$

Method 1: Chapter 3!

this is undamped forced oscillation at unnatural frequency!

x_H

x_p

$x_p + x_H$, match I.C.'s.

Method 2:

$$x'' + 4x = 10 \cos 3t$$

iff

$$\mathcal{L}\{x''(t) + 4x(t)\}(s) = \mathcal{L}\{10\cos 3t\}(s)$$

i.e.

$$s^2 X(s) - s\underset{\uparrow \atop 2}{x(0)} - \underset{\uparrow \atop 1}{x'(0)} + 4X(s) = 10\frac{s}{s^2+9}$$

$$X(s)(s^2+4) = 10\frac{s}{s^2+9} + 2s + 1$$

$$X(s) = 10\frac{s}{(s^2+9)(s^2+4)} + \frac{2s}{s^2+4} + \frac{1}{s^2+4}$$

done in Laplace land!
now use partial fracs & table to invert Laplace $X(s)$ to get $x(t)$

$$\frac{1}{(s^2+9)(s^2+4)} = \frac{1}{5}\left(\frac{1}{s^2+4} - \frac{1}{s^2+9}\right)$$

So

$$X(s) = \frac{2s}{s^2+4} - \frac{2s}{s^2+9} + \frac{2s}{s^2+4} + \frac{1}{s^2+4}$$

$$X(s) = \frac{4s}{s^2+4} + \frac{1}{s^2+4} - \frac{2s}{s^2+9}$$

$$\boxed{x(t) = 4\cos 2t + \tfrac{1}{2}\sin 2t - 2\cos 3t}$$

$f(t)$	$F(s)$	
1	$1/s$	$\text{Re}\, s > 0$
e^{at}	$1/(s-a)$	$\text{Re}\, s > a$
$\cos kt$	$s/(s^2+k^2)$	
$\sin kt$	$\frac{k}{s^2+k^2}$	
$f'(t)$	$sF(s) - f(0)$	
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	
$f'''(t)$	$s^3F(s) - s^2f(0) - sf'(0) - f''($	
$\cosh kt$	$s/(s^2-k^2)$	
$\sinh kt$	$k/(s^2-k^2)$	
$\vdots$		

(see book cover!)

$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$	
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$	
$\vdots$		

Use

$$\mathcal{L}\{e^{(a+ki)t}\}(s) = \frac{1}{s-(a+ki)}$$

and take real & imag parts.