Math 2280-1
Friday November 21
§6.3 Ecological models

(These are the §6.3 notes - it may take more than one lecture to finish them.)

Due December 1
6.3 8, 9, 10, 14, 15, 16, 17; in 8 and 10 create a pplane phase portrait for the nonlinear system (3), and explain how your linearization computations are reflected in the non-linear behavior near the corresponding equilibria.
6.4 12, 13, 14, 15, 16;

Due November 24
6.1 5, 8, 11, 15, 20, 24; use pplane to visualize your work in this section, as the directions indicate. However, you don't need to hand in any pplane hardcopies from this section.
6.2 5, 6, 7, 8, 9, 14, 15, 19, 27, 30. On 19, 27, use pplane to find and classify the other equilibrium solutions besides the origin, and print out and hand in the phase portrait justification (with sample solution trajectories). (For fun, you may wish to linearize about these other equilibria, to see how your pplane picture near the equilibrum corresponds to the solutions to the linearized system of DEs.)

- predator-prey:

$$PP \quad \begin{cases} x'(t) = ax - pxy = x(a-py) & \text{prey} \\ y'(t) = -by + qxy = y(-b+qx) & \text{predator} \end{cases} \qquad a, b, p, q > 0$$

natural region of interest: $x \geq 0,\ y \geq 0$

equil sol'ns:

$x = 0 \Downarrow y = 0$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$y = a/p \Downarrow x = \frac{b}{q}$, $\begin{bmatrix} b/q \\ a/p \end{bmatrix}$ in 1st quad. is this equil stable?

linearize:

$$\begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} ax - pxy \\ -by + qxy \end{bmatrix}$$

$$\text{Jacobian matrix} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} a - py & -px \\ qy & -b + qx \end{bmatrix}$$

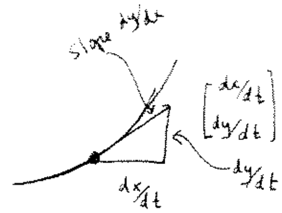
$$\text{at } \vec{x}_*, \quad J = \begin{bmatrix} 0 & -pb/q \\ aq/p & 0 \end{bmatrix}$$

$$|J - \lambda I| = \lambda^2 + ab \quad ; \quad \lambda = \pm i\sqrt{ab}$$

linearization DE is stable center (ellipse trajectories)

this is borderline for the original non-linear system....

method for understanding soln trajectories:



1210 Chain rule $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

So $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ (use this in §6.1 #24 HW too!)

for PP, $\frac{dy}{dx} = \frac{y(-b+qx)}{x(a-py)}$ separable!

$$\frac{a-py}{y}\,dy = \frac{-b+qx}{x}\,dx$$

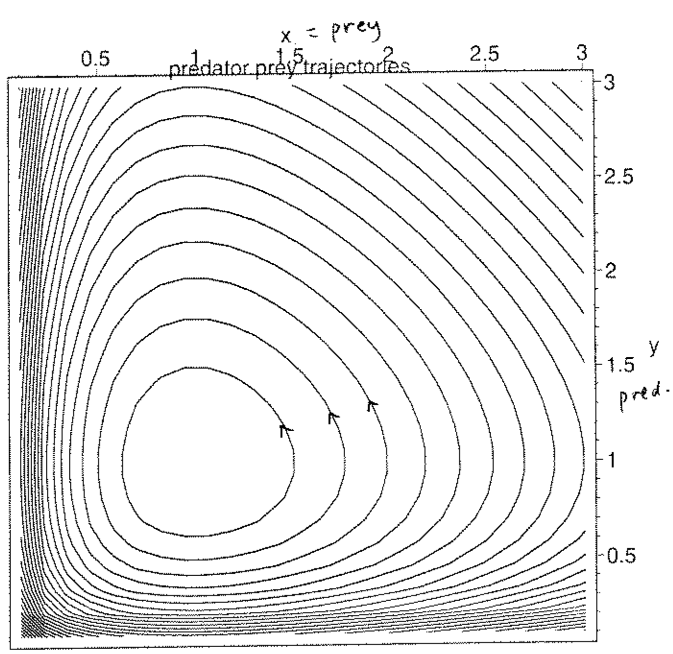
$$a\ln y - py = -b\ln x + qx + C \quad \text{(in } 1^{st} \text{ quadrant!)}$$

$$\underbrace{a\ln y + b\ln x - py - qx}_{f(x,y)} = C$$

So solution trajectories follow level curves of $f(x,y)$ (See figures 6.3.1 & 6.3.2 page 401)
and this $f(x,y)$ has a local max at $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b/q \\ a/p \end{bmatrix} = \vec{x}^*$

So $\vec{x}^*$ is stable, and general sol'ns are periodic.
- This oscillatory behavior is (apparently) seen in many real-world predator-prey systems.

```
> plot3d(ln(y)+ln(x)-y-x,x=0.1..3,y=0.1..3,
  style=contour,color=black,axes=boxed,orientation=[-90,0],
  contours=20,
  title='predator prey trajectories');
```



- competition model.

$$\frac{dx}{dt} = \underbrace{a_1 x - b_1 x^2}_{\text{logistic}} \underbrace{- c_1 xy}_{\text{competition}}$$

all params > 0

$$\frac{dy}{dt} = \underbrace{a_2 y - b_2 y^2}_{\text{logistic}} - \underbrace{c_2 xy}_{\text{competition?}}$$

any guesses?

critical points:

$$x\,[a_1 - b_1 x - c_1 y] = 0$$
$$y\,[a_2 - b_2 y - c_2 x] = 0$$

$x = 0$

$\Downarrow$

$y(a_2 - b_2 y) = 0$

$y = 0$ or $y = a_2/b_2$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 0 \\ a_2/b_2 \end{bmatrix}$

$x \neq 0$

$\Downarrow$

$a_1 - b_1 x - c_1 y = 0$

$y = 0$ or $a_2 - b_2 y - c_2 x = 0$

$y = 0 \Downarrow x = \frac{a_1}{b_1}$

$\begin{bmatrix} a_1/b_1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} b_1 & c_1 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

(book typo page 396 (8))

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{b_1 b_2 - c_1 c_2} \begin{bmatrix} b_2 & -c_1 \\ -c_2 & b_1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

call this $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$

$$= \frac{1}{b_1 b_2 - c_1 c_2} \begin{bmatrix} a_1 b_2 - c_1 b_1 \\ -a_1 c_2 + b_1^2 \end{bmatrix}$$

[this may or may not be in 1^{st} quadrant; depends on parameter values]

Text claims

1. If $c_1 c_2 < b_1 b_2$
   (relatively low competition between species vs. self-inhibitions)
   then $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$ is asymptotically stable equil, and $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to \begin{bmatrix} x_E \\ y_E \end{bmatrix}$ for any IVP if $x_0 > 0$, $y_0 > 0$.
2. if $c_1 c_2 > b_1 b_2$, $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$ is unstable, and either $x(t)$ or $y(t) \to 0$ as $t \to \infty$ (with 100% probability)

book shows examples - we'll work out the general proof!

proof:

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} a_1 - 2b_1 x - c_1 y & -c_1 x \\ -c_2 y & a_2 - 2b_2 y - c_2 x \end{bmatrix}$$

at $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$ we have $\begin{bmatrix} b_1 & c_1 \\ c_2 & b_2 \end{bmatrix}\begin{bmatrix} x_E \\ y_E \end{bmatrix} = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$

so (trickery!) at this point

$$J = \begin{bmatrix} -b_1 x_E & -c_1 x_E \\ -c_2 y_E & -b_2 y_E \end{bmatrix}$$

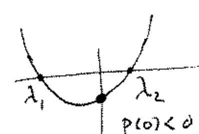
write $x = x_E$
$y = y_E$

$$p(\lambda) = |J - \lambda I| = (b_1 x + \lambda)(b_2 y + \lambda) - c_1 c_2 x y$$

$$p(\lambda) = \lambda^2 + (b_1 x + b_2 y)\lambda + (b_1 b_2 - c_1 c_2)\underbrace{xy}_{\substack{>0 \\ \text{by hypothesis}}}$$

the graph of $p(\lambda)$ is a concave up parabola

(2) if $c_1 c_2 > b_1 b_2$ then $p(0) < 0$
so $p(\lambda)$ has a positive and a negative root, by intermediate value thm, so $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$ is (unstable) saddle.

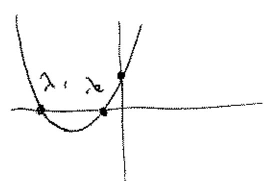


(1) if $c_1 c_2 < b_1 b_2$, then
vertex of parabola occurs at $\lambda = -\left(\frac{b_1 x + b_2 y}{2}\right)$ (complete square!)

and $p\left(-\frac{b_1 x + b_2 y}{2}\right) = \frac{1}{4}(b_1 x + b_2 y)^2 - \frac{1}{2}(b_1 x + b_2 y)^2 + (b_1 b_2 - c_1 c_2) x y$

$$= -\frac{1}{4}\Big[\underbrace{b_1^2 x^2 + b_2^2 y^2 + 2 b_1 b_2 x y \quad - 4 b_1 b_2 x y}_{(b_1 x - b_2 y)^2} + 4 c_1 c_2\Big]$$

$$< 0$$



since $p(0) > 0$ in this case,
$p(\lambda)$ has two negative roots.

$\blacksquare$

(this proves the stability claims for the linearized problem, not the global claims for the nonlinear problem)

Example with numbers: (See picture page 7 if you want to cheat)

Example 3 p397:

$$x' = 14x - \tfrac{1}{2}x^2 - xy = x\left(14 - \tfrac{1}{2}x - y\right)$$
$$y' = 16y - \tfrac{1}{2}y^2 - xy = y\left(16 - \tfrac{1}{2}y - x\right)$$

critical points

$$\begin{bmatrix}0\\0\end{bmatrix},\ \begin{bmatrix}0\\32\end{bmatrix},\ \begin{bmatrix}28\\0\end{bmatrix},\ \begin{bmatrix}12\\8\end{bmatrix}$$

We shall linearize at each critical point and then "guess" the phase portrait in the first quadrant.

$$J = \begin{bmatrix}F_x & F_y\\ G_x & G_y\end{bmatrix} = \begin{bmatrix}14 - x - y & -x\\ -y & 16 - y - x\end{bmatrix}$$

@ $\begin{bmatrix}0\\0\end{bmatrix}$ $\quad J = \begin{bmatrix}14 & 0\\ 0 & 16\end{bmatrix}$; nodal source $\quad \begin{bmatrix}u'\\v'\end{bmatrix} = \begin{bmatrix}14u\\16v\end{bmatrix}$

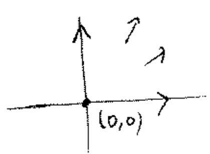
$\lambda = 14$ $\quad \lambda = 16$

$\vec{v} = \begin{bmatrix}1\\0\end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix}0\\1\end{bmatrix}$

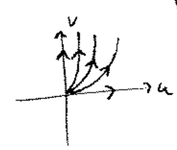
$$\frac{dv}{du} = \frac{v'}{u'} = \frac{8}{7}\frac{v}{u}$$

$$\frac{1}{v}\,dv = \frac{8}{7}\,\frac{1}{u}\,du \Rightarrow \ln|v| = \frac{8}{7}\ln|u| + C$$
$$v = Cu^{8/7}$$

rough:



more precise



@ $\begin{bmatrix}28\\0\end{bmatrix}$, $J = \begin{bmatrix}-14 & -28\\ 0 & -12\end{bmatrix}$

$\lambda = -12, -14$; nodal sink

$\vec{v} = \begin{bmatrix}14\\-1\end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix}1\\0\end{bmatrix}$

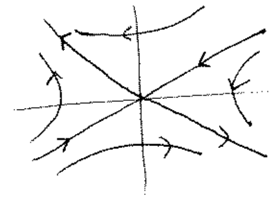
@ $\begin{bmatrix}0\\32\end{bmatrix}$, $J = \begin{bmatrix}-18 & 0\\ -32 & -16\end{bmatrix}$

$\lambda = -16, -18$ nodal sink

$\vec{v} = \begin{bmatrix}0\\1\end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix}1\\16\end{bmatrix}$

ⓐ $\begin{bmatrix}12\\8\end{bmatrix}$, $J = \begin{bmatrix}-6 & -12\\ -8 & -4\end{bmatrix}$

$|J-\lambda I| = \begin{vmatrix}-6-\lambda & -12\\ -8 & -4-\lambda\end{vmatrix}$



```
> with(linalg):
Warning, the protected names norm and trace have been redefined and unprotected
> A:=matrix(2,2,[-6,-12,-8,-4]);
```

$$A := \begin{bmatrix}-6 & -12\\ -8 & -4\end{bmatrix}$$

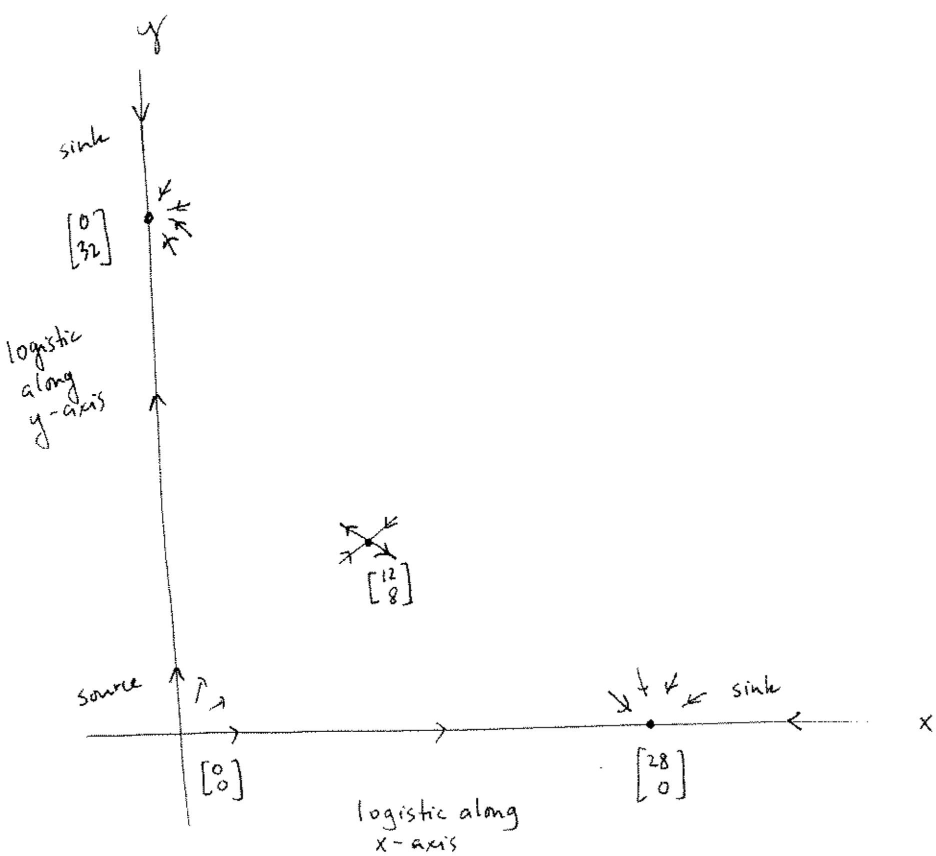
```
> eigenvectors(A);
```

$$\left[-5+\sqrt{97},\ 1,\ \left\{\left[\frac{1}{8}-\frac{\sqrt{97}}{8},\ 1\right]\right\}\right],\ \left[-5-\sqrt{97},\ 1,\ \left\{\left[\frac{1}{8}+\frac{\sqrt{97}}{8},\ 1\right]\right\}\right]$$

```
> evalf(eigenvectors(A));
```

[4.848857802, 1., {[-1.106107225, 1.]}], [-14.84885780, 1., {[1.356107225, 1.]}]

put it all together!! (then play with pplane)
(how's it doing with eigenvectors today?)



along $x=100$
$0 \le y \le 100$

$x' \le 14(100) - 5000 <$ [illegible]

along $y = 100$
$0 \le x \le 100$

$y' \le 16(100) - 5000 < 0$

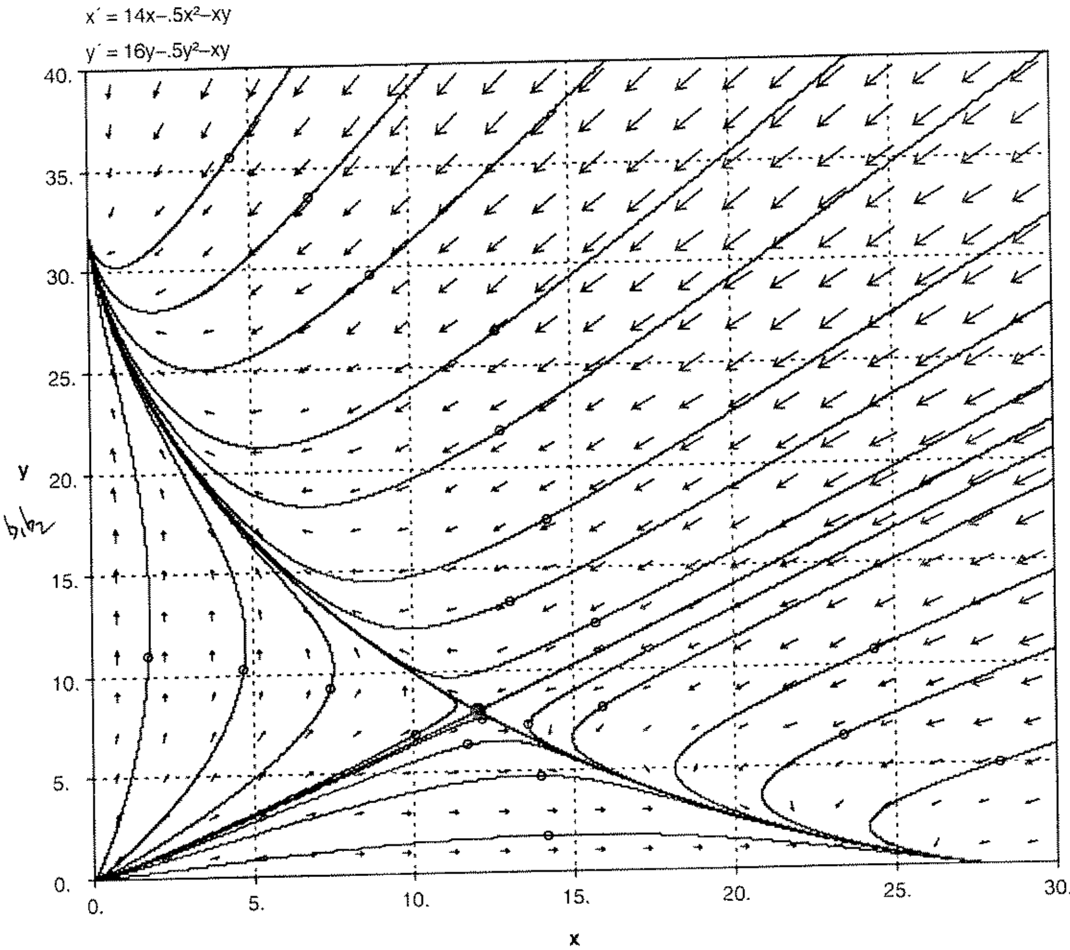
(huge populations decay

Use thm: only possibilities for sol'n traj's $\begin{bmatrix}x(t)\\y(t)\end{bmatrix}$ as $t \to \infty$ are

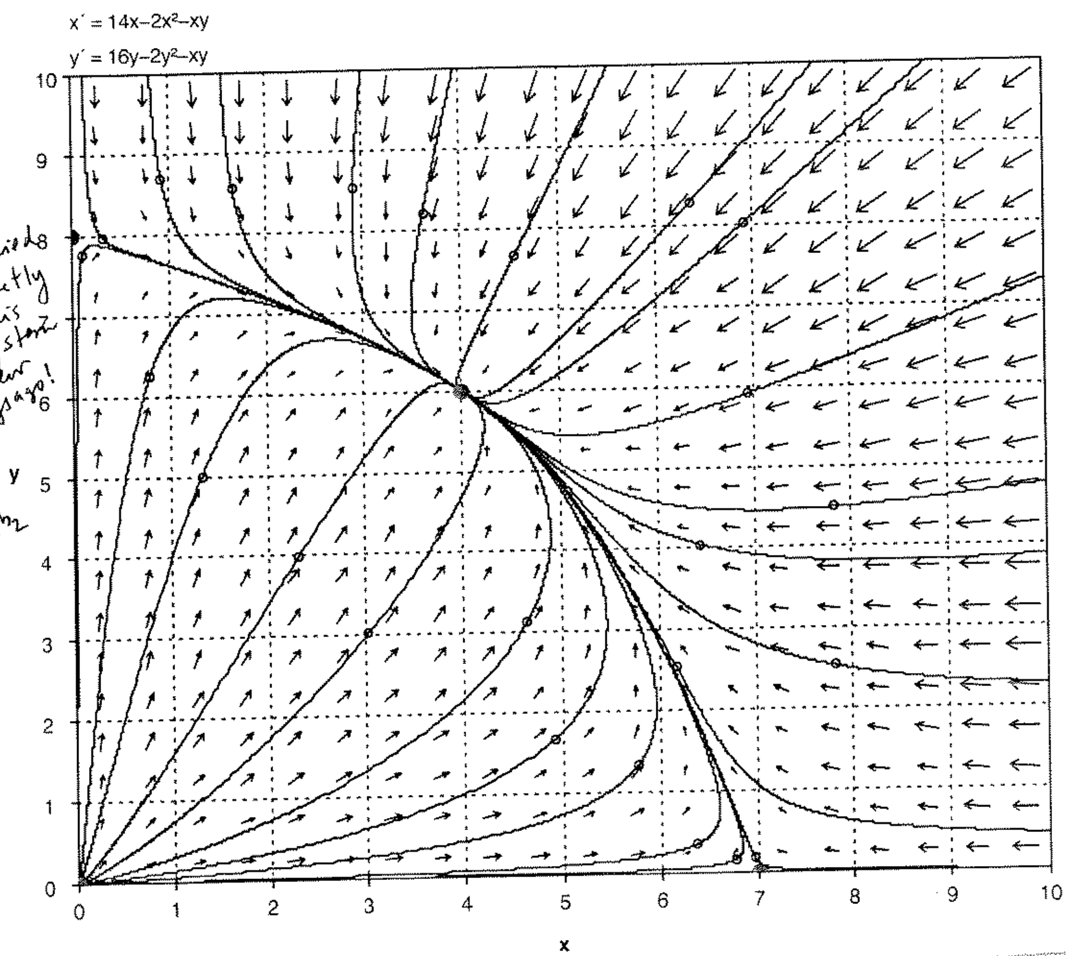
(1) $\begin{bmatrix}x(t)\\y(t)\end{bmatrix} \to \vec{x}_*$ equil

(2) $\begin{bmatrix}x(t)\\y(t)\end{bmatrix} \to \infty$

(3) $\begin{bmatrix}x(t)\\y(t)\end{bmatrix} \to$ periodic orbit



$c_1c_2 > b_1b_2$



We studied exactly this system a few days ago!

$c_1c_2 < b_1b_2$