Math 2280-1   6.2   – We'll do §6.3 ecological models on Friday. So postpone 6.3 HW to next assignment.

Wednesday November 19

Let's finish linearizing around the equilibria, in Tuesday's examples (pages 4, 5)

As we do so, we can discuss the general classification theorems for equilibrium solutions to autonomous linear and non-linear systems of 1st order DE's

(especially the case $n=2$)

Eigenvalues of A	Type of Critical Point
Real, unequal, same sign	Improper node
Real, unequal, opposite sign	Saddle point
Real and equal	Proper or improper node
Complex conjugate	Spiral point
Pure imaginary	Center

FIGURE 6.2.9. Classification of the critical point (0, 0) of the two-dimensional system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Linear

Eigenvalues λ_1, λ_2 for the Linearized System	Type of Critical Point of the Almost Linear System
$\lambda_1 < \lambda_2 < 0$	Stable improper node
$\lambda_1 = \lambda_2 < 0$	Stable node or spiral point
$\lambda_1 < 0 < \lambda_2$	Unstable saddle point
$\lambda_1 = \lambda_2 > 0$	Unstable node or spiral point
$\lambda_1 > \lambda_2 > 0$	Unstable improper node
$\lambda_1, \lambda_2 = a \pm bi \quad (a < 0)$	Stable spiral point
$\lambda_1, \lambda_2 = a \pm bi \quad (a > 0)$	Unstable spiral point
$\lambda_1, \lambda_2 = \pm bi$	Stable or unstable, center or spiral point

FIGURE 6.2.12. Classification of critical points of an almost linear system.

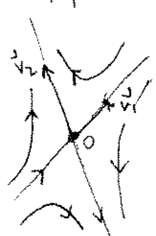
nonlinear – dominated by linearization

If λ real: $\vec{x}_H(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

or (if λ_1 is defective)

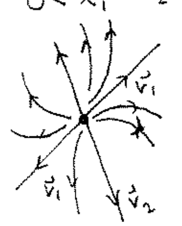
$\vec{x}_H(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 \left(e^{\lambda_1 t}\right)(t\vec{v}_1 + \vec{u})$

$\lambda_1 < 0 < \lambda_2$



$\boxed{\text{saddle}}$ (unstable)

$0 < \lambda_1 < \lambda_2$

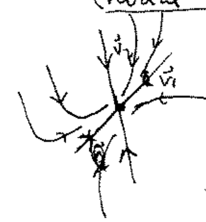


(unstable) $\boxed{\text{improper node}}$ (or nodal source)

$\lambda_1 = \lambda_2$ "proper"

$\lambda_2 < \lambda_1 < 0$

stable improper node (nodal sink)



$\lambda_1 = \lambda_2$ proper

If $\lambda = a + bi$ complex:

$a<0$ (stable) asy. spiral sink

$a>0$ (unstable) spiral source

$a=0$ (stable) center for linear; indeterminant for non-linear

<u>Theorem A</u> General n: Consider the DE $\vec{x}' = A_{n\times n}\vec{x}$. If the real part of each eigenvalue of A is negative then $x^* = \vec{0}$ is asymptotically stable equilibrium. If, on the other hand, any eigenvalue of A has positive real part then $\vec{x}^* = \vec{0}$ is unstable. If $\mathrm{Re}(\lambda_i) \le 0 \ \forall i$, and some $\mathrm{Re}\,\lambda_i = 0$ then $x^* = \vec{0}$ may or may not be stable, depending on whether or not the corresponding eigenspace(s) is defective.

<u>proof</u>: Can you fill this in using all of our work on linear systems of DE's ??

Theorem B (For non-linear equilibria) 2

If the linearization for $\vec{x}' = F(\vec{x})$ at the ~~sol~~ equil. sol'n $\vec{x}^*$ satisfies
$\vec{u}' = A\vec{u}$
and if all evals of A have strictly neg. real part, then $\vec{x}^*$ is asymptotically stable. If ~~at~~ some λ_i has positive real part then $\vec{x}^*$ is unstable. All other cases are borderline.
(the proof of this theorem requires analysis estimates)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

Complex eigenvalues details for page 1 classification

$\lambda = a \pm bi \qquad \vec{w} = \vec{u} \pm i\vec{v}$

$$A(\vec{u} + i\vec{v}) = (a+bi)(\vec{u} + i\vec{v})$$
$$A(\vec{u} - i\vec{v}) = (a-bi)(\vec{u} - i\vec{v})$$

$$\Rightarrow \begin{aligned} A\vec{u} &= a\vec{u} - b\vec{v} \\ A\vec{v} &= b\vec{u} + a\vec{v} \end{aligned}$$
(add eqtns, divide by 2)
(subt eqtns, divide by $2i$)

$$\Rightarrow [A]_{\{\vec{u},\vec{v}\}} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} := B$$ (a rotation-dilation matrix!)

$$S^{-1} A S = B \qquad S = [\vec{u} | \vec{v}].$$

$$\vec{x}' = A\vec{x}$$
$$\vec{x}' = SBS^{-1}\vec{x}$$
$$(S^{-1}\vec{x})' = B(S^{-1}\vec{x})$$; write $\begin{bmatrix} x \\ y \end{bmatrix}_{\{\vec{u},\vec{v}\}} = S^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} z \\ w \end{bmatrix}$

$$\begin{bmatrix} z' \\ w' \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}$$

sol'n!
$$\begin{bmatrix} z \\ w \end{bmatrix} = C_1 e^{at} \begin{bmatrix} \cos bt \\ -\sin bt \end{bmatrix} + C_2 e^{at} \begin{bmatrix} \sin bt \\ \cos bt \end{bmatrix}$$

$$\begin{bmatrix} z \\ w \end{bmatrix} = e^{at} \begin{bmatrix} C_1 \cos bt + C_2 \sin bt \\ C_1 \cos(bt + \pi/2) + C_2 \sin(bt + \pi/2) \end{bmatrix}$$

$$= e^{at} \begin{bmatrix} C \cos(bt - \alpha) \\ C \cos(bt + \pi/2 - \alpha) \end{bmatrix}$$

$$\begin{bmatrix} z(t) \\ w(t) \end{bmatrix} = C e^{at} \begin{bmatrix} \cos(bt - \alpha) \\ -\sin(bt - \alpha) \end{bmatrix}$$

circle radius C

spirals outward if $a > 0$
inward if $a < 0$

Finally, $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ is an ellipse if $a = 0$ else an elliptical spiral