Math 2280
Tuesday November 18

Linearization: Yesterday (last Wednesday's notes)

we linearized the DE

$$\frac{dx}{dt} = 14x - 2x^2 - xy \qquad \text{rabbits}$$
$$\frac{dy}{dt} = \underbrace{16y - 2y^2}_{\text{logistic}} - \underbrace{xy}_{\text{competition}} \qquad \text{squirrels}$$

near the interesting equilibrium solution $\begin{bmatrix}4\\6\end{bmatrix}$

We wrote
$$x = 4 + u$$
$$y = 6 + v$$
so that small $\begin{bmatrix}u\\v\end{bmatrix}$ corresponds to $\begin{bmatrix}x\\y\end{bmatrix}$ close to $\begin{bmatrix}4\\6\end{bmatrix}$.

We got
$$\begin{bmatrix}u'\\v'\end{bmatrix} = \underbrace{\underbrace{\begin{bmatrix}-8 & -4\\-6 & -12\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}}_{\text{linear piece}}}_{\text{this linear DE has}} + \underbrace{\begin{bmatrix}-4u^2-uv\\-2v^2-uv\end{bmatrix}}_{\text{error}}$$

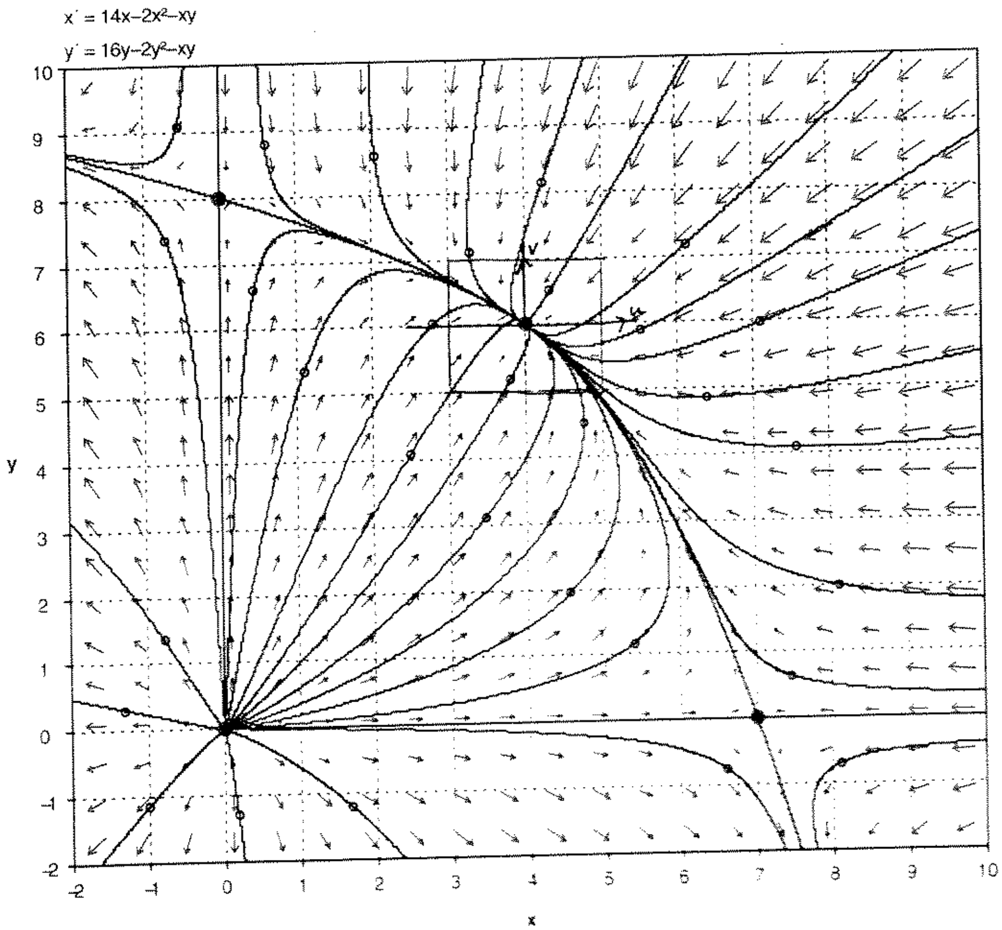
error; if $\left\|\begin{bmatrix}u\\v\end{bmatrix}\right\| < \underset{\text{tiny}}{\delta}$ then $\|\text{error}\| \le 8\underset{\text{tiny squared}}{\delta^2}$

this linear DE has
eigenvectors(A) = [-4.708.1, {[.77, -.63]}], [-15.3, 1 {[.49, .89]}

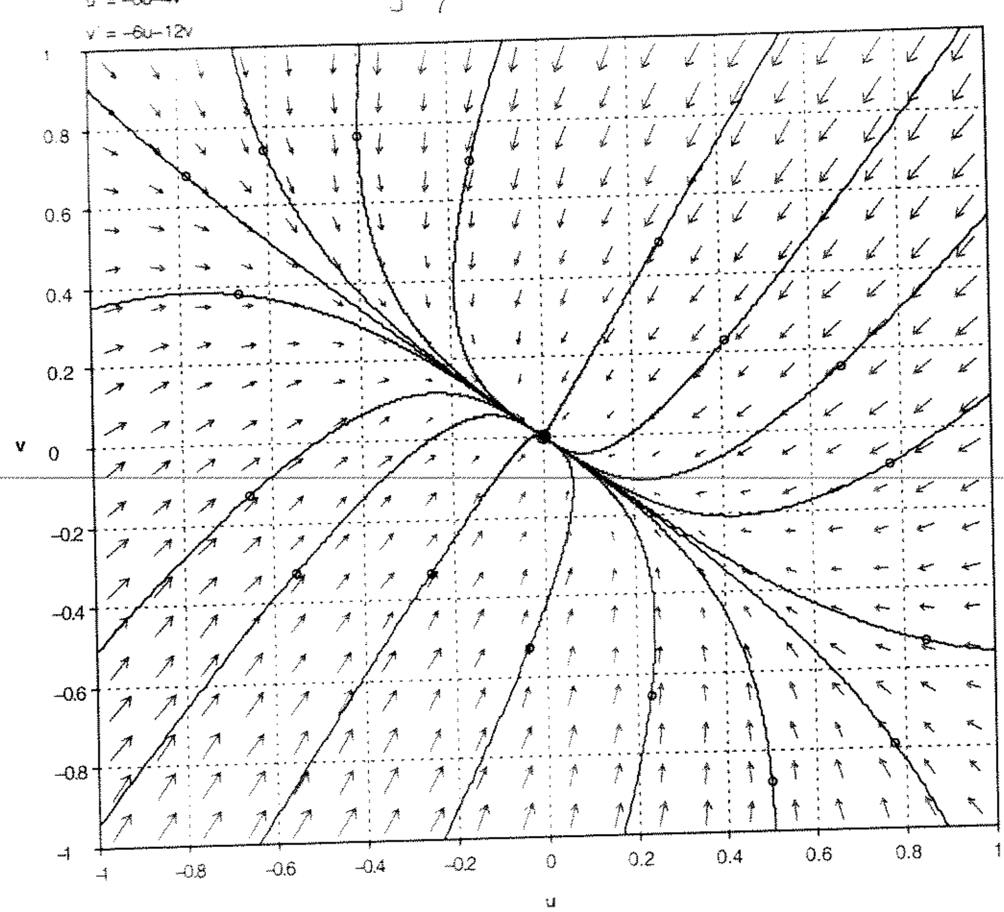
- On the next page compare the (blown-up) phase portrait for the original system, near $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}4\\6\end{bmatrix}$, to the linearized phase portrait.

- Find the eigenvectors in the linearized phase portrait, and explain the connection between the tangent vector field and the eigendata, using

$$\begin{bmatrix}u(t)\\v(t)\end{bmatrix} \cong c_1 e^{-4.7t}\begin{bmatrix}.77\\-.63\end{bmatrix} + c_2 e^{-15.3t}\begin{bmatrix}.49\\.89\end{bmatrix}$$



magnify & linearize!



Linearization (works for systems of n DE's; illustrated for $n=2$)

Let (1) $\begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$

$F(x_*, y_*) = F(P) = 0$
$G(x_*, y_*) = G(P) = 0$

write $x(t) = x_* + u(t)$
$y(t) = y_* + v(t)$

we are interested in what happens for $\|(u,v)\|$ small.

error; $\frac{\varepsilon}{\|(u,v)\|} \to 0$ as $(u,v)\to(0,0)$

$$x' = F(x_*+u, y_*+v) = \overbrace{F(x_*,y_*)}^{0} + F_x(x_*,y_*)u + F_y(x_*,y_*)v + \varepsilon_1(u,v)$$
$$y' = G(x_*+u, y_*+v) = \underbrace{G(x_*,y_*)}_{0} + G_x(-)u + G_y(-)v + \varepsilon_2(u,v)$$

Math 2210 affine approx.

$u' = x' = F_x u + F_y v + \varepsilon_1(u,v)$
$v' = y' = G_x u + G_y v + \varepsilon_2(u,v)$

where the partial derivs of F & G are evaluated at the equil. pt.

(2) $$\begin{bmatrix} u' \\ v' \end{bmatrix} = \underbrace{\begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}}_{\text{"}A\text{"}} \begin{bmatrix} u \\ v \end{bmatrix}$$

(Partial derivs in A are evaluated at (x^*, y^*))

this is the linearization of (1), at (x^*, y^*).
the eigenvector data of A determines stability for the nonlinear system (1), in the non borderline cases.

the matrix A is called the Jacobian matrix for $\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$, at $\begin{bmatrix} x_* \\ y_* \end{bmatrix}$

• Exercise For the system on page 1,
Check that the Jacobian matrix yields the same linearization that we got "the long way"

- <u>Exercise</u> Compute the linearizations of our rabbit-squirrel model at the three other equilibrium solns. Compare to phase portrait and eigendata

partial answer: The Jacobian matrix at $\begin{bmatrix} x \\ y \end{bmatrix}$ is

$$J(x,y) = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 14-4x-y & -x \\ -y & 16-4y-x \end{bmatrix}$$

We'll classify the equilibrium points of linear systems tomorrow, and it's a deep theorem that these classifications essentially carry over for equilibria in non-linear autonomous systems.

For today, find the equilibria for $x' = x - y - x^2 + xy$ (6.1 HW # 8)
$y' = -y - x^2$

and linearize about the one in the 3rd quadrant (complex eigenvectors!)

$$J = \begin{bmatrix} 1-2x+y & -1+x \\ -2x & -1 \end{bmatrix}$$