Math 2280-1
Tuesday Nov. 11
Bonus day!

Jordan Canonical form, "chains", and e^{At} when A is not diagonalizable.

Recall from 2270

- If V is a finite dimensional vector space with basis $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\} = \beta$ and if $v = \sum_{i=1}^{n} c_i \vec{v}_i$ then $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is called the coord vector for $\vec{v}$, wrt β; we write $[\vec{v}]_\beta = \vec{c}$.

  If $L: V \to V$ is linear the matrix "$[L]_\beta$" for L wrt β has the property that
  $$[L]_\beta [v]_\beta = [Lv]_\beta$$
  so $col_j [L]_\beta = [Lv_j]_\beta$

  If $\{w_1 \ldots w_n\} = \gamma$ is another basis for V, then
  $$[L]_\gamma = S^{-1} [L]_\beta S$$
  where S is the change of basis matrix, $S = S_{\beta \leftarrow \gamma}$ i.e. $S[v]_\gamma = [v]_\beta$

- If A, B are matrices, they are similar iff $B = S^{-1}AS$ for some invertible matrix S.
  two matrices are similar iff they represent the same linear transformation, but with respect to two different bases.

  Since eigenspaces (and generalized eigenspaces) can be defined in terms of the operator L rather than the particular matrix representing it, geometric data such as
  $\dim(\ker(L - \lambda_j I)^\ell)$,
  $\dim(\text{range}(L - \lambda_j I)^\ell)$
  can be computed using any matrix which represents L.

- Our convention will be to use A to represent $L\vec{x} = A\vec{x}$ with respect to the standard basis $\{\vec{e}_1, \vec{e}_2, \ldots \vec{e}_n\}$ and to use other letters for similar matrices $B = S^{-1}AS$, which represent $[L]_{\{\text{cols of } S\}}$.

Theorem 1 Let $|A-\lambda I| = (-1)^n (\lambda-\lambda_1)^{k_1}\cdots(\lambda-\lambda_m)^{k_m}$ $\quad$ $\lambda_i \neq \lambda_j$ unless $i=j$, $\sum_{j=1}^{n} k_j = n$

- Then the generalized eigenspace
$$G_{\lambda_j} = \ker (A-\lambda_j I)^{k_j}$$
has $\dim(G_{\lambda_j}) = k_j$. Furthermore, $A : G_{\lambda_j} \to G_{\lambda_j}$.
- Amalgamating bases of each G_{λ_j} yields a basis of $\mathbb{C}^n$.

So the matrix of A with respect to this amalgamated basis is block diagonal

$$[A]_\beta = \begin{array}{c} \begin{array}{ccccc} k_1 & k_2 & & & k_m \end{array} \\ \begin{bmatrix} B_1 & & & & O \\ & B_2 & & & \\ & & \square & & \\ & & & \square & \\ O & & & & B_m \end{bmatrix} \end{array}$$

$B_j = [A]$ restricted to G_{λ_j}, wrt a G_{λ_j} basis

Lemma 1 Let $\{\vec{v}_1, \vec{v}_2, .. \vec{v}_\ell\}$ be a chain of length ℓ, i.e.

$$(A-\lambda_j I) v_\ell = v_{\ell-1}$$
$$(A-\lambda_j I)^2 v_\ell = (A-\lambda_j I) v_{\ell-1} = v_{\ell-2}$$
$$\vdots$$
$$(A-\lambda_j I) v_2 = v_1 \neq 0$$
$$(A-\lambda_j I)^\ell v_\ell = (A-\lambda_j I) v_1 = 0$$

Then $\{v_1, v_2, .. v_\ell\}$ are linearly independent.

proof: $c_1 v_1 + c_2 v_2 + .. + c_\ell v_\ell = 0$ $\quad \Rightarrow c_1 = 0$

$(A-\lambda_j I)$: $\quad c_2 v_1 + c_3 v_2 + \cdots + c_\ell v_{\ell-1} = 0$ $\quad \cdots$

$\vdots$ $\quad \Rightarrow c_{\ell-1} = 0$

$(A-\lambda_j I)^{\ell-1}$: $\quad c_\ell v_1 = 0 \quad \Rightarrow c_\ell = 0$ ∎

Lemma 2 If $(A-\lambda_j I)^\ell v = 0$ for some v, then in fact $(A-\lambda_j I)^{k_j} v = 0$, i.e. $v \in G_{\lambda_j}$.

proof If $\ell \le k_j$ there is nothing to show. So assume $\ell > k_j$.
set $v = v_\ell$, and construct the chain $\{v_1, v_2 .. v_\ell\}$ as in Lemma 1.
Complete this to a basis for $\mathbb{C}^n$, $\{v_1, .. v_\ell, w_{\ell+1}, .. w_n\} =: \beta$

Then $$[A]_\beta = \begin{array}{c} \ell\left\{\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right. \\ \\ \end{array} \left[\begin{array}{cccc|c} \lambda_j & 1 & & O & \\ & \lambda_j & 1 & & \\ & & \ddots & 1 & \text{mess} \\ O & & & \lambda_j & \\ \hline & & O & & \end{array}\right]$$

so $|[A]_\beta - \lambda I| = |A - \lambda I|$
$\qquad \| $
$(\lambda-\lambda_j)^\ell \, p_{n-\ell}(\lambda)$

Thus $\ell \le k_j$ since the $(\lambda-\lambda_j)$ factor in the charact poly for A has power k_j ∎

proof of theorem 1. Let $\lambda_j = \lambda_1$ by choice of numbering.

Let $V = \ker (A-\lambda_1 I)^{k_1} = G_{\lambda_1}$
$W = \text{range}\,(A-\lambda_1 I)^{k_1}$

Let $\dim V = s$. Then by rank + nullity theorem, $\dim W = n-s$

We claim

- $A: V \to V$ proof: if $(A-\lambda_1 I)^{k_1} v = 0$ then $(A-\lambda_1 I)^{k_1} Av = A (A-\lambda_1 I)^{k_1} v = A0 = 0.$
- $A: W \to W$ proof: if $w = (A-\lambda_1 I)^{k_1} v$ then $Aw = A (A-\lambda_1 I)^{k_1} v = (A-\lambda_1 I)^{k_1} (Av) \in \text{range}\,(A-\lambda_1 I)^{k_1}$
- $V \cap W = \{0\}$ proof: Let $v_1 = (A-\lambda_1 I)^{k_1} v_2$ with $v_1, v_2 \in V$. We show $v_1 = 0$:

$(A-\lambda_1 I)^{k_1} v_1 = 0$ by def of V
so $(A-\lambda_1 I)^{2k_1} v_2 = 0$.
so $\underbrace{(A-\lambda_1 I)^{k_1} v_2}_{= v_1} = 0$ by lemma 2 ■

Therefore, if $\{v_1, v_2, \ldots v_s\}$, $\{w_1, w_2, \ldots w_{n-s}\}$ are bases for V, W
then $\{v_1, v_2, \ldots v_s, w_1, w_2, \ldots w_{n-s}\}$ is an independent set
so is a basis for $\mathbb{C}^n$.
$\left(c_1 v_1 + \ldots + c_s v_s = -d_1 w_1 - d_2 w_2 \ldots - d_{n-s} w_{n-s} \Rightarrow \text{RHS} = 0 = \text{LHS} \text{ by } V \cap W = \{0\} \Rightarrow c_i = 0,\ d_i = 0\ \forall i,j \right)$

Let B be the matrix for A w.r.t. this basis. Then by 1st 2 bullets,

$$B = \begin{array}{c} \begin{matrix} \overbrace{}^{s} & \overbrace{}^{n-s} \end{matrix} \\ \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{matrix} \}s \\ \}n-s \end{matrix} \end{array}$$

similar matrices have same charact. polys

$$|B-\lambda I| = |B_1 - \lambda I|\,|B_2 - \lambda I| = |A - \lambda I| = (\lambda-\lambda_1)^{k_1} \text{(other factors w/o } (\lambda-\lambda_1) \text{ terms)}$$

$|B_1 - \lambda I|$: this must be $(-1)^s (\lambda-\lambda_1)^s$
because if $\vec{c} = [v]_{\{v_1 \ldots v_s\}}$ then $[Av] = B_1 \vec{c}$
and so if $B_1 \vec{c} = \lambda \vec{c}$
then $[Av] = \lambda [v]$
so $Av = \lambda v$
so $A - \lambda_1 v = (\lambda - \lambda_1) v$
and $\underbrace{(A-\lambda_1 I)^k v}_{\substack{\text{must} = 0 \\ \text{by def of } V}} = (\lambda-\lambda_1)^k v$
So every eval of $B_1 = \lambda_1$.

Also, λ_1 cannot be a root of $|B_2 - \lambda I|$ since then V would be missing at least 1 λ_1-eigenvector

Thus, by uniqueness of factorization

$$s = k_1 \ !$$

So $\dim G_{\lambda_1} = k_1$, $A: G_{\lambda_1} \to G_{\lambda_1}$ from first bullet
Now apply induction to the matrix B_2! to get the block structure (true when $n=1$)

Chain structure within G_{λ_j} blocks:

<u>Theorem 2</u> The basis for G_{λ_j} may be chosen in chains, so that the basis is a union of all the vectors in all the chains. The number of chains and their lengths are uniquely determined by the values of

$$\dim\left(\ker(A-\lambda_j I)^s\right) \qquad s=1,2,\dots k_j$$

} the matrix of A w.r.t. the chained basis of each G_{λ_j} is called the <u>Jordan canonical form</u>

<u>proof</u>: We exhibit an algorithm to construct the claimed chain structure. Consider a block matrix A_1, for the restriction of A to G_{λ_1}, as on previous page. Write $B = A_1 - \lambda_1 I$.

Then $\ker B \subsetneq \ker B^2 \subsetneq \cdots \subsetneq \ker B^s = \ker B^{s+1} = \cdots = \ker B^{k_1} = G_{\lambda_1}$

dimensions $d_1 < d_2 < \quad < d_s = k_1$ $\uparrow$ after 1^{st} $=$, so are the rest: if $B^{s+1}v=0 \Rightarrow B^s v = 0$ then $B^{s+2}v = 0 = B^{s+1}Bv \Rightarrow B^s Bv = 0 \Rightarrow B^s v = 0$ etc.

So $\quad \operatorname{range} B \supsetneq \operatorname{range} B^2 \supsetneq \cdots \supsetneq \operatorname{range} B^s = 0$

dimensions $\quad k_1 - d_1 > k_1 - d_2 > \cdots > 0$

<u>Algorithm</u>: www.numbertheory.org/jordan.html

1. pick basis $\{u_1, v_1, \dots w_1\}$ for range B^{s-1}
complete backwards to length s chains:

$$\begin{array}{ccc}
u_s & v_s \cdots & w_s \\
\downarrow & \downarrow & \downarrow \\
u_{s-1} & v_{s-1} & w_{s-1} \\
\downarrow B & \vdots & \downarrow \\
u_2 & v_2 & \vdots \\
\downarrow B & \downarrow & \downarrow \\
u_1 & v_1 & w_1
\end{array}$$

This collection is independent because the bottom row is, so if you set a linear combo of all $=0$, apply $B^{s-1} \Rightarrow$ top row coeffs $=0$ then apply $B^{s-2} \Rightarrow$ next row coeffs $=0$ etc.

2. the vectors $\begin{array}{cccc} u_2 & v_2 & & w_2 \\ \downarrow & \downarrow & \cdots & \downarrow \\ u_1 & v_1 & & w_1 \end{array}$ $\; r_1 \cdots s_1$ are in range B^{s-2}. ($\dim = k - d_{s-2}$) If they're a basis, proceed to step 3. If not, ~~add~~ complete to a basis by adding
and complete backwards to chains of length $s-1$:

So, now:
$$\begin{array}{ccccc}
u_s & v_s & w_s & & \\
\downarrow & \downarrow & \downarrow & & \\
u_{s-1} & v_{s-1} & w_{s-1} & r_{s-1} & s_{s-1} \\
\downarrow & \downarrow & \vdots & \downarrow & \downarrow \\
\vdots & \vdots & & r_{s-2} & s_{s-2} \\
 & & & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \vdots & \\
u_1 & v_1 \cdots & w_1 & r_1 \cdots & s_1
\end{array}$$

<u>This</u> collection is lin ind, same reason

3. the first three rows are in range B^{s-3}. If they're a basis, go to step 4. If not, complete to a basis by adding more vectors to bottom row.

<u>Induct!</u> Tableaux finishes after you add eigenvectors which are not in the range of B^1; these are your length 1 chains

Example Suppose $|A-\lambda I| = (\lambda-2)^{11}(\lambda+3)^{3}$

		range dims		range dims
with	$\dim(\ker(A+3I)) = 2$	1	$\dim(\ker(A-2I)) = 4$	7
	$\dim(\ker(A+3I)^2) = 3$	0	$\dim(\ker(A-2I)^2) = 8$	3
			$\dim(\ker(A-2I)^3) = 10$	1
			$\dim(\ker(A-2I)^4) = 11$	0

Find the Jordan canonical form of A!

Find the matrix of e^{At} with respect to the chained (Jordan) basis!

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ans: $J = S^{-1}AS = \begin{bmatrix} \boxed{-3} & & & & & \\ & \boxed{\begin{matrix}-3 & 1\\ 0 & -3\end{matrix}} & & & 0 & \\ & & \begin{matrix}2&1&0&0\\0&2&1&0\\0&0&2&1\\0&0&0&2\end{matrix} & & & \\ & & & \begin{matrix}2&1&0\\0&2&1\\0&0&2\end{matrix} & & \\ & 0 & & & \begin{matrix}2&1\\0&2\end{matrix} & \\ & & & & & \begin{matrix}2& \\0&2\end{matrix} \end{bmatrix}$

$$[e^{At}]_{\substack{\text{Jord}\\ \text{basis}}} = \begin{bmatrix} \boxed{e^{-3t}} & & & \\ & e^{-3t}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix} & & \\ 0 & & e^{2t}\begin{bmatrix}1 & t & t^2/2 & t^3/6\\ 0 & 1 & t & t^2/2\\ 0 & 0 & 1 & t\\ 0 & 0 & 0 & 1\end{bmatrix} & \\ & & & \text{etc.} \end{bmatrix}$$

(so $e^{At} = S[e^{At}]_{JB}S^{-1}$)

Playing with Jordan canonical form **and** *matrix exponentials*
#39 section 5.5

Notice how quickly we figure out what the Jordan form of A is, and then that it's somewhat more work to come up with a chain basis to transform A into that Jordan form.

```
> with(linalg) :
> A := matrix(4, 4, [1, 3, 3, 3,
                     0, 1, 3, 3,
                     0, 0, 2, 3,
                     0, 0, 0, 2]);
  eigenvectors(A);
```

$$A := \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\left[1, 2, \left\{\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}\right]\right\}\right], \left[2, 2, \left\{\left[\begin{array}{cccc} 12 & 3 & 1 & 0 \end{array}\right]\right\}\right] \quad (1)$$

```
>
```

We know the chain structure for this small matrix immediately from the eigenvectors result! Thus we know the Jordan canonical form J without even finding the actual chains, and we know that exponential (At) is similar to exponential(Jt), which is easy to compute. But since we've got Maple running, we may as well find the chain bases:

```
> iden := matrix(4, 4, [1, 0, 0, 0,
                        0, 1, 0, 0,
                        0, 0, 1, 0,
                        0, 0, 0, 1]);
```

$$iden := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2)$$

```
> kernel(A - iden);
  kernel( (A - iden)^2);
```

$$\left\{\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}\right]\right\}$$

$$\left\{\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \end{array}\right]\right\} \quad (3)$$

```
> u2 := matrix(4, 1, [0, 1, 0, 0]);
  u1 := evalm((A - iden)&*u2); #chain of length 2
```

$$u2 := \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$u1 := \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (4)$$

> $kernel(A - 2 \cdot iden);$
> $kernel((A - 2 \cdot iden)^2);$

$$\{[\,12 \;\; 3 \;\; 1 \;\; 0\,]\}$$

$$\{[\,-51 \;\; -6 \;\; 0 \;\; 1\,], [\,12 \;\; 3 \;\; 1 \;\; 0\,]\} \qquad (5)$$

> $v2 := matrix(4, 1, [-51, -6, 0, 1]);$
> $v1 := evalm((A - 2 \cdot iden) \& \cdot v2);$ *#chain of length 2*

$$v2 := \begin{bmatrix} -51 \\ -6 \\ 0 \\ 1 \end{bmatrix}$$

$$v1 := \begin{bmatrix} 36 \\ 9 \\ 3 \\ 0 \end{bmatrix} \qquad (6)$$

> $S := augment(u1, u2, v1, v2);$ *#change of basis matrix*

$$S := \begin{bmatrix} 3 & 0 & 36 & -51 \\ 0 & 1 & 9 & -6 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (7)$$

> $J := evalm(S^{(-1)} \& \cdot A \& \cdot S);$ *#Jordan canonical form explicitly as a similar matrix*

$$J := \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad (8)$$

> $exponential(J, t);$

$$\begin{bmatrix} e^t & t\,e^t & 0 & 0 \\ 0 & e^t & 0 & 0 \\ 0 & 0 & e^{2t} & t\,e^{2t} \\ 0 & 0 & 0 & e^{2t} \end{bmatrix} \tag{9}$$

\> *evalm*$\left(S\&\cdot exponential(J,t)\&\cdot S^{(-1)}\right)$; *#these will be the same*
exponential(A,t);

$$\begin{bmatrix} e^t & 3\,t\,e^t & -12\,e^t - 9\,t\,e^t + 12\,e^{2t} & 51\,e^t + 18\,t\,e^t + 36\,t\,e^{2t} - 51\,e^{2t} \\ 0 & e^t & -3\,e^t + 3\,e^{2t} & 6\,e^t + 9\,t\,e^{2t} - 6\,e^{2t} \\ 0 & 0 & e^{2t} & 3\,t\,e^{2t} \\ 0 & 0 & 0 & e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} e^t & 3\,t\,e^t & -12\,e^t - 9\,t\,e^t + 12\,e^{2t} & 51\,e^t + 18\,t\,e^t + 36\,t\,e^{2t} - 51\,e^{2t} \\ 0 & e^t & -3\,e^t + 3\,e^{2t} & 6\,e^t + 9\,t\,e^{2t} - 6\,e^{2t} \\ 0 & 0 & e^{2t} & 3\,t\,e^{2t} \\ 0 & 0 & 0 & e^{2t} \end{bmatrix} \tag{10}$$

\>