Math 2280-1
Monday Nov. 10

Variation of parameters in action:

$$* \quad \vec{x}'(t) = P(t)\vec{x} + \vec{f}(t)$$

$\Phi(t)$ FMS for $\vec{x}' = P(t)\vec{x}$

Write any soltn to * as

$\vec{x}(t) = \Phi\vec{u}$ (i.e. $\vec{u} = \Phi^{-1}\vec{x}$).

plug into *

$\underbrace{\Phi' u}_{P\Phi} + \Phi u' = P\Phi u + f$ , so $\vec{x}$ solves * iff $\Phi u' = f$

iff $u' = \Phi^{-1} f$ ; $u = \int \Phi^{-1}\vec{f}\, dt$.

p. 368

<u>Exercise 1</u> Show that converting the second order linear DE

$$y''(t) + P(t)y' + Q(t)y = f(t)$$

into a first order system for $x_1 = y$, $x_2 = y'$

recovers the mysterious §3.5 variation of parameters formula as a special case of this one.
(or, if you dare, do the $n^{th}$ order linear DE)

p.367
Exercise 2 

Solve
$$\begin{cases} \vec{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \vec{x} - \begin{bmatrix} 15 \\ 4 \end{bmatrix} t e^{-2t} \\ \vec{x}(0) = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \end{cases}$$

use the var par formula for the case $A$ = const matrix, $\Phi(t) = e^{At}$ :

$$e^{-tA}(\vec{x}' - A\vec{x}) = e^{-At}\vec{f}(t)$$

$$(e^{-At}\vec{x})' = \vec{0}$$

$$\int_0^t : \quad e^{-At}\vec{x}(t) - \vec{x}_0 = \int_0^t e^{-As}\vec{f}(s)ds$$

$$\vec{x}(t) = e^{At}\vec{x}_0 + e^{At}\int_0^t e^{-As}\vec{f}(s)ds$$

Math 2280-1
November 10, 2008
nonhomogeneous linear systems of differential equations -
variation of parameters

Work for example 4 page 367

```
> with(linalg):
> A:=matrix(2,2,[4,2,3,-1]);
  eigenvectors(A);
  exponential(A,s);
```

$$A := \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$[5, 1, \{[2, 1]\}], [-2, 1, \{[1, -3]\}]$$

$$\begin{bmatrix} \frac{1}{7}e^{(-2s)} + \frac{6}{7}e^{(5s)} & \frac{2}{7}e^{(5s)} - \frac{2}{7}e^{(-2s)} \\ \frac{3}{7}e^{(5s)} - \frac{3}{7}e^{(-2s)} & \frac{6}{7}e^{(-2s)} + \frac{1}{7}e^{(5s)} \end{bmatrix}$$

if you were using "undetermined coeffs", how would you guess $\vec{x}_p$?

$e^{As}$ !

```
> x0:=matrix(2,1,[7,3]);
```

$$x0 := \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

```
> f:=t->evalm(-t*exp(-2*t)*matrix(2,1,[15,4]));
  #the nonhomogeneous term.
```

$$f := t \to \mathrm{evalm}(-t\, e^{(-2t)}\, \mathrm{matrix}(2, 1, [15, 4]))$$

```
> f(s);
```

$$\begin{bmatrix} -15\, s\, e^{(-2s)} \\ -4\, s\, e^{(-2s)} \end{bmatrix}$$

```
> evalm(exponential(-A,s)&*f(s));
```

$$\begin{bmatrix} -15\left(\frac{6}{7}e^{(-5s)} + \frac{1}{7}e^{(2s)}\right) s\, e^{(-2s)} - 4\left(-\frac{2}{7}e^{(2s)} + \frac{2}{7}e^{(-5s)}\right) s\, e^{(-2s)} \\ -15\left(-\frac{3}{7}e^{(2s)} + \frac{3}{7}e^{(-5s)}\right) s\, e^{(-2s)} - 4\left(\frac{1}{7}e^{(-5s)} + \frac{6}{7}e^{(2s)}\right) s\, e^{(-2s)} \end{bmatrix}$$

```
> simplify(%);
```

$$\begin{bmatrix} -s\,(14\, e^{(-7s)} + 1) \\ -s\,(-3 + 7\, e^{(-7s)}) \end{bmatrix}$$

← compare p. 367

Maple (or I) currently has trouble directly evaluating the solution formula: I can't seem to make Maple integrate matrix valued functions without writing an explicit procedure (this did not used to be the case in older versions): The procedure below takes a matrix valued expression in s and returns a matrix expression in t, for which each entry is the integral of the input from s=0 to s=t:

```
> Integratematrix:=proc(mat,m,n,s)
  local H, #matrix function to be returned
        i,  #row index
        j;  #column index

  H:=matrix(m,n):
   for i from 1 to m do
     for j from 1 to n do
     H[i,j]:=int(mat[i,j],s=0..t):
     od:
   od:
  return(evalm(H));
  end:
> integrand:=simplify(evalm(exponential(-A,s)&*f(s)));
     #the integrand in the solution formula
```

$$integrand := \begin{bmatrix} -s\,(14\, e^{(-7s)} + 1) \\ -s\,(-3 + 7\, e^{(-7s)}) \end{bmatrix}$$

← p 367-368

```
> Integratematrix(integrand,2,1,s);
```

$$\begin{bmatrix} 2\, t\, e^{(-7t)} + \frac{2}{7}e^{(-7t)} - \frac{t^2}{2} - \frac{2}{7} \\ \frac{3\, t^2}{2} + t\, e^{(-7t)} + \frac{1}{7}e^{(-7t)} - \frac{1}{7} \end{bmatrix}$$

```
> sol:=t->evalm(exponential(A,t)&*(x0+Integratematrix(integrand,2,1,
  s)));
     #solution formula
```

$$sol := t \to \mathrm{evalm}(\text{`\&*`}(\mathrm{exponential}(A, t), x0 + \mathrm{Integratematrix}(integrand, 2, 1, s)))$$

```
> simplify(sol(t));  #see page 368!!!
```

$$\begin{bmatrix} \frac{3}{7}e^{(-2t)} - \frac{1}{2}e^{(-2t)}\, t^2 + \frac{46}{7}e^{(5t)} + 2\, e^{(-2t)}\, t \\ \frac{23}{7}e^{(5t)} + e^{(-2t)}\, t - \frac{2}{7}e^{(-2t)} + \frac{3}{2}e^{(-2t)}\, t^2 \end{bmatrix}$$

```
>
```