

Name.....Solutions.....
I.D. number.....

Math 2280-1
Second Midterm
November 14, 2008

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work and justify your conclusions.** There are 100 points possible, and the point values for each problem are indicated in the right-hand margin.
Good Luck!

1) Let A be the matrix

$$A := \begin{bmatrix} 0 & 1 \\ -29 & -4 \end{bmatrix}.$$

Maple says

$$\begin{bmatrix} > \text{eigenvectors}(A); \\ [-2 + 5I, 1, \{[1 \ -2 + 5I]\}], [-2 - 5I, 1, \{[1 \ -2 - 5I]\}] \end{bmatrix} \quad (1)$$

(Remember that Maple uses capital I instead of i for the square root of -1.)

1a) Using this information, find the general solution to the homogeneous first order system

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -29 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (1)$$

$$\begin{aligned} e^{\lambda t} \vec{v} &= e^{(-2+5i)t} \begin{bmatrix} 1 \\ -2+5i \end{bmatrix} \\ &= e^{-2t} (\cos 5t + i \sin 5t) \begin{bmatrix} 1 \\ -2+5i \end{bmatrix} \\ &= e^{-2t} \begin{bmatrix} \cos 5t \\ -2\cos 5t - 5\sin 5t \end{bmatrix} + i e^{-2t} \begin{bmatrix} \sin 5t \\ -2\sin 5t + 5\cos 5t \end{bmatrix} \end{aligned}$$

real $e^{\lambda t} \vec{v}$
imag $e^{\lambda t} \vec{v}$
solns

$$\vec{x}(t) = c_1 e^{-2t} \begin{bmatrix} \cos 5t \\ -2\cos 5t - 5\sin 5t \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} \sin 5t \\ -2\sin 5t + 5\cos 5t \end{bmatrix} \quad (10 \text{ points})$$

1b) Carefully explain the connection between the first order system in 1a) and the second order linear differential equation

$$\frac{d^2}{dt^2} y(t) + 4 \left(\frac{d}{dt} y(t) \right) + 29 y(t) = 0. \quad (2)$$

Then write down the general solution to this DE, using your work from (1a). (Of course, you might want to check that your answer is right using the theory we developed for homogeneous linear differential equations, or secretly use that method to check your answer to 1a.)

they are equivalent, under the identification (10 points)

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t). \end{aligned}$$

(since then

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= y'' = -4y' - 29y \\ &= -29x_1 - 4x_2. \end{aligned}$$

Thus the general soln to (2) is the 1st component to the general soln of 1, i.e.

$$y(t) = c_1 e^{-2t} \cos 5t + c_2 e^{-2t} \sin 5t$$

(check: $y = e^{rt}$, $p(r) = r^2 + 4r + 29 = 0$
 $(r+2)^2 + 25 = 0$
 $r+2 = \pm 5i$ ✓)

1c) Find the general solution to

$$\frac{d^2}{dt^2} y(t) + 4 \left(\frac{d}{dt} y(t) \right) + 29 y(t) = 52 \cos(5t).$$

Identify the steady periodic and transient parts of the general solution.

(10 points)

$$\begin{aligned} + 29(y_p) &= A \cos 5t + B \sin 5t \\ + 4(y_p') &= -5A \sin 5t + 5B \cos 5t \\ 1(y_p'') &= -25A \cos 5t - 25B \sin 5t \end{aligned}$$

$$\begin{aligned} \text{want} &= \cos 5t [52] \\ L y_p &= \cos 5t [(29-25)A + 20B] \\ &+ \sin 5t [-20A + (29-25)B] \end{aligned}$$

$$\begin{bmatrix} 4 & 20 \\ -20 & 4 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 52 \\ 0 \end{bmatrix}$$

$$A = \frac{\begin{vmatrix} 52 & 0 \\ 0 & 4 \end{vmatrix}}{16 + 400} = \frac{208}{416} = \frac{1}{2}$$

$$B = \frac{\begin{vmatrix} 4 & 52 \\ -20 & 0 \end{vmatrix}}{416} = \frac{1040}{416} = \frac{520}{208} = \frac{260}{104} = \frac{130}{52} = \frac{10}{4} = \frac{5}{2} !$$

$$y(t) = \frac{1}{2} \cos 5t + \frac{5}{2} \sin 5t + c_1 e^{-2t} \cos 5t + c_2 e^{-2t} \sin 5t$$

steady periodic

transient

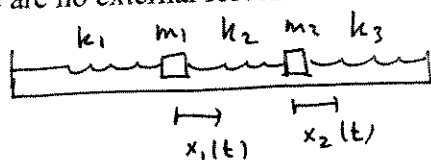
1d) Use your work in the previous sections to write down the general solution to the system

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -29 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 52 \cos(5t) \end{bmatrix}$$

(5 points)

$$\begin{aligned} \vec{x}(t) &= \begin{bmatrix} \frac{1}{2} \cos 5t + \frac{5}{2} \sin 5t \\ -\frac{5}{2} \sin 5t + \frac{25}{2} \cos 5t \end{bmatrix} + c_1 e^{-2t} \begin{bmatrix} \cos 5t \\ -5 \sin 5t - 2 \cos 5t \end{bmatrix} \\ &+ c_2 e^{-2t} \begin{bmatrix} \sin 5t \\ 5 \cos 5t - 2 \sin 5t \end{bmatrix} \end{aligned}$$

2) Consider a two mass, 3 spring system consisting of two masses with mass m_1 and m_2 , and springs with Hooke's constants k_1, k_2, k_3 , as sketched below. You can think of the masses as sliding along a frictionless table, or as hanging vertically. Let $x_1(t)$ and $x_2(t)$ measure displacement from equilibrium, as indicated. Assume there are no external forces.



2a) Derive the system of differential equations which governs the motion of the masses. (5 points).

$$m_1 x_1'' = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$m_2 x_2'' = -k_3 x_2 - k_2 (x_2 - x_1)$$

or, collecting terms into matrix form

$$\begin{bmatrix} m_1 x_1'' \\ m_2 x_2'' \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 - k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2b) What is the dimension of the solution space to this system of DEs? Explain briefly. (5 points)

4-d: 2nd order linear homog DE's are equivalent to a homog. linear sys of 4 1st order DE's, for $\begin{bmatrix} x_1 \\ x_1' \\ x_2 \\ x_2' \end{bmatrix}$, so general theory implies 4-d soln space.

2c) In case each mass is 2 kg and the spring constants are each 4 N/m, show that the system in (2a) reduces to (5 points)

$$\begin{bmatrix} \frac{d^2}{dt^2} x_1(t) \\ \frac{d^2}{dt^2} x_2(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$2x_1'' = -8x_1 + 4x_2$$

$$2x_2'' = 4x_1 - 8x_2$$

÷ 2 in eqs

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -4x_1 + 2x_2 \\ 2x_1 - 4x_2 \end{bmatrix}$$

2d) Find the general solution to the mass-spring system in (2c), and explain the two fundamental motions (modes) which superpose to yield this general solution. For your convenience, the system is rewritten below.

$$\begin{bmatrix} \frac{d^2}{dt^2} x_1(t) \\ \frac{d^2}{dt^2} x_2(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

(15 points)

$$\begin{vmatrix} -4-\lambda & 2 \\ 2 & -4-\lambda \end{vmatrix} = (\lambda+4)^2 - 4 = (\lambda+4+2)(\lambda+4-2) = (\lambda+6)(\lambda+2)$$

$\lambda = -6, -2$ evals

$\lambda = -6 :$

$$\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}$$

$$\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\omega = \sqrt{6}$$

$\lambda = -2$

$$\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \\ \hline 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}$$

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\omega = \sqrt{2}$$

$$\vec{x}(t) = (c_1 \cos \sqrt{6}t + c_2 \sin \sqrt{6}t) \begin{bmatrix} -1 \\ 1 \end{bmatrix} + (c_3 \cos \sqrt{2}t + c_4 \sin \sqrt{2}t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



fast mode, $\omega = \sqrt{6}$
 $T = \frac{2\pi}{\sqrt{6}}$

masses oscillate with
 equal amplitude but
 in opposite dir's
 ("out of phase")



slow mode, $\omega = \sqrt{2}$
 $T = \frac{2\pi}{\sqrt{2}}$

masses oscillate with
 equal amplitude and in
 parallel ("in phase").

3a) Find the matrix exponential e^{Bt} where B is the matrix from problem 2,

$$B := \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix}.$$

You may refer to eigenvalue-eigenvector calculations you did in that problem without redoing them.

(10 points)

$$\Phi(t) = \left[e^{-6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} -e^{-6t} & e^{-2t} \\ e^{-6t} & e^{-2t} \end{bmatrix}$$

$$\Phi(0) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \Phi(0)^{-1} = \frac{1}{-2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$e^{At} = \Phi(t) \Phi(0)^{-1} = \frac{1}{2} \begin{bmatrix} -e^{-6t} & e^{-2t} \\ e^{-6t} & e^{-2t} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{-6t} + e^{-2t} & -e^{-6t} + e^{-2t} \\ -e^{-6t} + e^{-2t} & e^{-6t} + e^{-2t} \end{bmatrix}$$

3b) Find the matrix exponential e^{Ct} , where C is the matrix

$$C := \begin{bmatrix} -2 & 0 \\ 6 & -2 \end{bmatrix}.$$

(Hint: the eigenspace for the eigenvalue -2 is defective.)

$$C = -2I + N, \quad N = \begin{bmatrix} 0 & 0 \\ 6 & 0 \end{bmatrix}, \quad N^2 = \begin{bmatrix} 0 & 0 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(10 points)

$$e^{Ct} = e^{-2It + Nt}$$

$$= e^{-2It} e^{Nt} \quad \text{because } I \& N \text{ commute}$$

$$= e^{-2t} I e^{Nt}$$

$$= e^{-2t} [I + Nt + 0] \quad \leftarrow \text{all other terms are zero.}$$

$$e^{Ct} = e^{-2t} \begin{bmatrix} 1 & 0 \\ 6t & 1 \end{bmatrix}$$

3c) Let A be any matrix. Explain why the two formulas for e^{At} , namely the power series definition and the product of any fundamental matrix solution with its inverse at $t=0$, must yield the same answer.

(5 points)

$$\text{Both } e^{At} := I + tA + \frac{t^2}{2!} A^2 + \dots = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

$$\text{and } e^{At} := \Phi(t) \Phi(0)^{-1}$$

are the unique soln to the matrix IVP

$$\begin{cases} Z' = AZ \\ Z(0) = I \end{cases}$$

So they must be equal!

$$\left(\text{since } \text{col}_j(Z) \text{ is the unique soln to } \begin{cases} \frac{dx}{dt} = Ax \\ x(0) = \vec{e}_j \end{cases} \right).$$

4) Consider a first order system initial value problem

$$\frac{dx}{dt} = Ax + f(t)$$

$$x(0) = x_0$$

where A is a (constant) n by n matrix, and $x(t)$ is a vector valued function. Derive an integral formula for the solution to this system using matrix exponentials. Justify your steps.

(10 points)

$$x'(t) - Ax = f(t)$$

$$e^{-At} (x' - Ax) = e^{-At} f(t)$$

(e^{-At} invertible)

$$(e^{-At} x)' = e^{-At} f(t)$$

product rule and $(e^{-At})' = -Ae^{-At} = e^{-At}(-A)$

$\int_0^t :$

$$e^{-At} x(t) - x_0 = \int_0^t e^{-As} f(s) ds$$

$$e^{-At} x(t) = x_0 + \int_0^t e^{-As} f(s) ds$$

$$\boxed{x(t) = e^{At} x_0 + e^{At} \int_0^t e^{-As} f(s) ds}$$