Name $\mathcal{S}$olutions

I.D. number

Math 2280-1

Exam 1

October 3, 2008

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin.

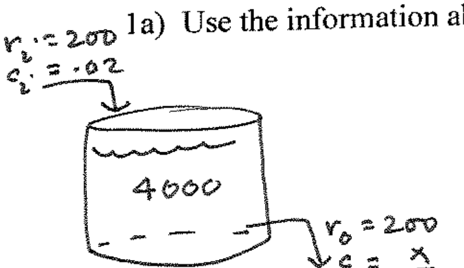
Good Luck!

1) Consider a brine tank which holds 4000 gallons of continuously-mixed liquid. Let x(t) be the amount of salt (in pounds) in the tank at time t. The in-flow and out-flow rates are both 200 gallons/hour, and the concentration of salt flowing into the tank is 0.02 pounds per gallon.

1a) Use the information above and your modeling ability to derive the differential equation for x(t):

$$\frac{dx}{dt} = 4 - \frac{1}{20}x$$

(5 points)



$$\frac{dx}{dt} = r_i c_i - r_o c_o$$

$$\frac{dx}{dt} = 200(.02) - 200\left(\frac{x}{4000}\right) \qquad \frac{200}{4000} = \frac{1}{20}$$

$$\boxed{\frac{dx}{dt} = 4 - \frac{x}{20}}$$

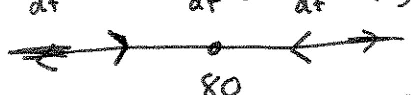
1b) What is the equilibrium solution to this differential equation? Is it stable or unstable? Explain using a phase portrait.

(5 points)

$$\frac{dx}{dt} = 0 = 4 - \frac{x}{20} \qquad \frac{x}{20} = 4 \qquad \boxed{x = 80} \text{ lb.}$$

$$\frac{dx}{dt} = -\frac{1}{20}(x - 80)$$

$\frac{dx}{dt} > 0$ $\quad \frac{dx}{dt} = 0$ $\quad \frac{dx}{dt} = (\text{neg})(\text{pos}) = \text{neg}$



$\boxed{x = 80 \text{ is stable}}$

(asymptotically stable)

1c) Find the solution to the initial value problem for this differential equation,

$$\frac{dx}{dt} = 4 - \frac{1}{20}x,$$

assuming there are originally 20 pounds of salt in the tank. Use either the Chapter 1 or Chapter 3 methods for linear differential equations.

(10 points)

$$\frac{dx}{dt} + .05x = 4$$

$$e^{.05t}(x' + .05x) = 4e^{.05t}$$

$$e^{.05t}x = \int 4e^{.05t}\,dt = 80e^{.05t} + C$$

$$x = 80 + Ce^{-.05t}$$

$$x(0) = 20 = 80 + C \quad \text{so} \quad C = -60$$

$$\boxed{x = 80 - 60e^{-.05t}}$$

1d) Is $\lim_{t\to\infty} x(t)$ consistent with your discussion in (1b)? How could your smart little sister have figured out the limiting amount of salt in the tank without understanding any differential equations, i.e. just by using the in-flow concentration?

(5 points)

$\lim_{t\to\infty} x(t) = 80$ since $e^{-.05t} \to 0$ as $t \to \infty$.

Consistent with 1b, since phase portrait indicates $\lim_{t\to\infty} x(t) = 80$ for <u>every</u> IVP sol'n.

Sister says that tank concentration will eventually be (nearly) incoming concentration of .02 lb/gall

thus amt $= (.02)\ \text{lb/gal} \cdot 4000\ \text{gal}$

$= 80$ lb.

2a) The identical differential equation to the one you just studied in the first problem,

$$\frac{dx}{dt} = 4 - \frac{1}{20}x,$$

could also represent a Newton's law of cooling model for the temperature x(t) of an object being heated or cooled in an ambient medium. Rewrite the DE how we wrote it in this context, and identify the ambient temperature in this case.

(5 points)

$$\frac{dx}{dt} = k(A-x)$$

$$\frac{dx}{dt} = \frac{1}{20}(80-x) \qquad \text{so } \boxed{A = 80^\circ}$$

2b) It was our habit to solve Newton's law of cooling differential equations using separable DE techniques. So, now, solve the initial value problem for this differential equation that way, assuming the initial temperature of the object is 20 degrees. (You've got your answer from (1c) to compare with!)

(10 points)

$$\frac{dx}{x-80} = -\frac{1}{20}\,dt$$

$$\ln|x-80| = -\frac{1}{20}t + C_1$$

$$|x-80| = e^{C_1} e^{-\frac{1}{20}t}$$

$$x - 80 = Ce^{-\frac{1}{20}t}$$

$$x = 80 + Ce^{-\frac{1}{20}t}$$

$$x(0) = 20 \Rightarrow C = -60$$

$$\boxed{x = 80 - 60e^{-\frac{1}{20}t}}$$

or, $\frac{dv}{dt} = 4 - \underbrace{\frac{1}{20}v}_{\text{drag}}$

so $\underbrace{300\frac{dv}{dt}}_{\text{mass}\cdot\text{accel}} = \underbrace{1200}_{\text{const accel}} - 15v$

2c) This identical differential equation could also arise if "x" (which we'd usually write as "v") stands for the velocity of an object subject to a constant acceleration force as well as to a drag force proportional to velocity. If the mass of the object was 300 kg, and we're using meters and seconds, what would be the value of the constant acceleration force? What is the object's terminal velocity in this case? Include units for both answers.

$$m\frac{dv}{dt} = C - \ell v$$

(5 points)

$$\frac{dv}{dt} = \frac{1}{20}(80-v)$$

$$\boxed{V_T = 80 \text{ m/s}}$$

$$\frac{dv}{dt} = \frac{C}{m} - \frac{\ell}{m}v = 4 - \frac{1}{20}v$$

$m = 300$

$$\frac{C}{300} = 4 \qquad \boxed{C = 1200 \text{ N}}$$

3) Consider the initial value problem

$$x''(t) + 4\,x'(t) + 29\,x(t) = 0$$
$$x(0) = 1$$
$$x'(0) = \cancel{16}\ -7$$

which could arise in a damped mass- spring problem.

3a) Find the general solution to the differential equation. What type of damping is exhibited here?

(7 points)

$x = e^{rt}$

$p(r) = r^2 + 4r + 29 = (r+2)^2 + 25 = (r+2+5i)(r+2-5i) = 0$

$r = -2 \pm 5i$

$e^{rt} = e^{-2t} e^{5it}$ one sol

$= e^{-2t}\cos 5t$

$+ i e^{-2t}\sin 5t$ another

soltn ~~bas~~

$$\boxed{x_H = e^{-2t}(A\cos 5t + B\sin 5t)}$$

underdamped

3b) Solve the initial value problem.

(10 points)

$x(0) = \boxed{1 = A}$

$x'(0) = -\cancel{16}7 = -2A + 5B$

$-5 = 5B \;;\; \boxed{B = -1}$

$x' = e^{-2t}\begin{bmatrix} -2A\cos & -2B\sin \\ -5A\sin & +5B\cos \end{bmatrix}$

at $t=0$,

$x'(0) = -2A + 5B$

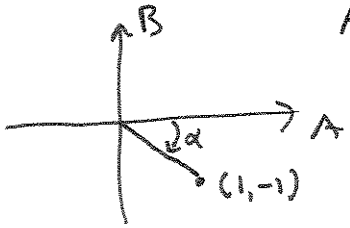
$$\boxed{x(t) = e^{-2t}(\cos 5t - \sin 5t)}$$

3c) Rewrite your solution so that the trigonometric part is in phase-amplitude form. If you made no errors the phase is an "elementary" angle that you don't need your calculator to compute.

(8 points)

$A\cos\omega t + B\cos\omega t = C\cos(\omega t - \alpha)$

$A = 1,\; B = -1 \qquad C = \sqrt{A^2+B^2} = \sqrt{2}$



$\frac{B}{A} = \arctan\alpha$

$-1 = \arctan\alpha$

$\alpha = -\pi/4$

$$\boxed{x(t) = e^{-2t}\sqrt{2}\cos(5t + \pi/4)}$$

4a) What is the dimension of the space of solutions y(x) to the differential equation

$$y'''(x) + y''(x) - 5y'(x) + 3y(x) = 0 ?$$

(3 points)

dim = order = 3

4b) Find a basis for the solution space in 4a. (Hint: r=1 is one root of the characteristic polynomial.)

(10 points)

$y = e^{rx}$

$p(r) = r^3 + r^2 - 5r + 3 = 0$

$p(1) = 1+1-5+3 = 0$

$$\begin{array}{r|l} & r^2 + 2r - 3 \\ \hline r-1 & r^3 + r^2 - 5r + 3 \\ & r^3 - r^2 \\ \hline & 2r^2 - 5r \\ & 2r^2 - 2r \\ \hline & -3r + 3 \\ & -3r + 3 \\ \hline & 0 \end{array}$$

$r^2 + 2r - 3 = (r+3)(r-1)$

so $p(r) = (r-1)^2(r+3)$

$\{e^x, xe^x, e^{-3x}\}$

4c) Prove the functions you found in you part 4b basis are linearly independent. (You may perhaps like to use the Wronskian or related ideas.)

(7 points)

$$\det(W) = \begin{vmatrix} e^x & xe^x & e^{-3x} \\ e^x & e^x(x+1) & -3e^{-3x} \\ e^x & e^x(x+2) & 9e^{-3x} \end{vmatrix}$$

$$@\ x=0,\ W = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & -3 \\ 1 & 2 & 9 \end{vmatrix} = 9 + 2 - 1 + 6 = 16$$

since $W \neq 0$ the fns e^x, xe^x, e^{-3x} are lin ind.

5a) Write down Euler's formula, i.e. the formula which relates exponentials to trigonometric functions.

(5 points)

$$e^{i\theta} = \cos\theta + i\sin\theta$$

5b) Use Euler's formula to show that the rule of exponents,

$$e^{i(\alpha+\beta)} = e^{i\alpha} e^{i\beta}$$

is equivalent to the addition angle formulas for sine and cosine.

(5 points)

$$e^{i(\alpha+\beta)} = e^{i\alpha} e^{i\beta}$$

$$\Leftrightarrow \quad \cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$
$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$$

$\Leftrightarrow$ real parts agree & imag parts agree, i.e.

$$\left.\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ \sin(\alpha+\beta) &= \cos\alpha\sin\beta + \sin\alpha\cos\beta\end{aligned}\right\}$$ these are the addition angle formulas