Math 2280-1
Monday December 8

§9.3-9.4

Fourier Series with period 2L

- − do page 3 Friday's notes

§9.3
We've noticed that

<u>odd</u> 2L-periodic fcns'
Fourier series are <u>sine series</u>,
because all $a_n = 0$.

<u>even</u> 2L-periodic fcns'
Fourier series are <u>cosine series</u>
because all $b_n = 0$.

So, If $f: [0,L] \to \mathbb{R}$

- you can extend $f: [-L,L] \to \mathbb{R}$ as an even fcn $f(t) := f(-t)$, and then to $\mathbb{R}$ as a 2L-periodic fcn $\left(f(t+2kL) := f(t) \quad -L<t<L\right)$

$$\Rightarrow f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L} n t\right)$$

<u>cosine series for f</u>

Or,

- you can extend $f: [-L,L] \to \mathbb{R}$ as an odd fcn $f(-t) := f(t) \quad 0<t<L$, and then to $\mathbb{R}$ as a 2L-periodic fcn

$$\Rightarrow f \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L} n t\right)$$

<u>sine series for f</u>.

(on the interval $0<t<L$ either series will converge to $f(t)$, wherever f is differentiable!)

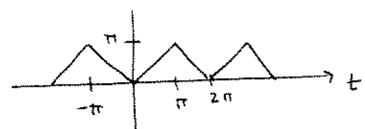
(Useful in §9.5-9.6)

HW for Friday Dec. 12

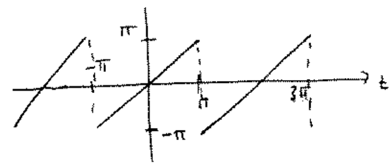
9.3 (1, 9, 17, 19, 20)
9.4 (1, 7, 9, 13)
9.5 1 2 7
9.6 1 5

} we'll work most of these in class, and §9.5, 9.6 not on final exam.

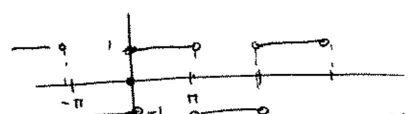
<u>Our zoo</u>



$$\text{tent}(t) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos nt}{n^2}$$



$$\text{sawtooth}(t) \sim 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nt}{n}$$



$$\text{square}(t) \sim \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n}$$

If f is 2L-periodic

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L} n t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L} n t\right)$$

$$\langle f, g \rangle = \frac{1}{L} \int_{-L}^{L} f(t) g(t)\, dt$$

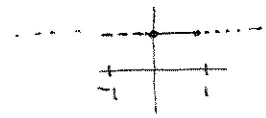
$$a_0 = \langle f, 1 \rangle = \frac{1}{L} \int_{-L}^{L} f(t)\, dt$$

$$a_n = \langle f, \cos \tfrac{\pi}{L} n t \rangle = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{\pi}{L} n t\right) dt$$

$$b_n = \langle f, \sin \tfrac{\pi}{L} n t \rangle = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{\pi}{L} n t\right) dt$$

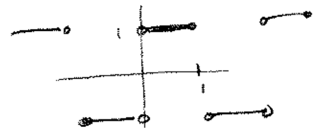
examples (we can steal)

1. let $f(t) = 1$, $0 \le t \le 1$



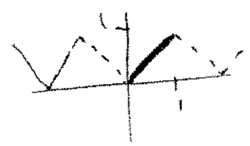
What is the cosine series for f?

What is the sine series for f?

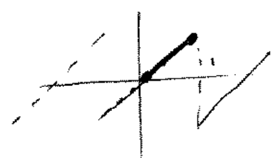


2. let $g(t) = t$, $0 \le t \le 1$

What is the cosine series for g?



What is the sine series for g?



We talked about differentiating Fourier series term by term.
There is also:

<u>Theorem</u> : If f is p.w. cont., $2L$-periodic, with Fourier series (page 605)

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right)$$

then the antiderivative

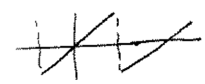
$$\int_0^t f(s)\,ds = \frac{a_0 t}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{L}{n\pi}\right) \sin\left(\frac{n\pi}{L} t\right) + b_n \left(\frac{L}{n\pi}\right)\left[-\cos\left(\frac{n\pi}{L} t\right) + 1\right]$$

integrated term by term.

series on the right converges to this antideriv!

<u>Example</u> (9.3 #19)

start with $t = \text{sawtooth}(t) \sim 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nt}{n}$ $\qquad -\pi < t < \pi$



Integrate once to deduce

$$\frac{t^2}{2} = 2 \sum_{n=1}^{\infty} (-1)^n \frac{\cos nt}{n^2} + \frac{\pi^2}{12}$$

(In HW need to get all the way to $\frac{t^4}{24}$!)

<u>Hint</u> : Use $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$

Fourier & springs (§9.4)

Let $f(t)$ be $2L$-periodic,
consider

$$x''(t) + \omega_0^2 x = f(t)$$

$x_H = A\cos\omega_0 t + B\sin\omega_0 t.$

so $T_0 =$ "natural period" $= \frac{2\pi}{\omega_0}$.

But when we played "guess the resonance" we saw that even if f's period $2L \neq T_0$ you may get resonance. (Conversely, even if f has period $\frac{2\pi}{\omega_0}$ you might not get resonance).

Here's the answer.

$$f \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\tfrac{\pi}{L} n t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\tfrac{\pi}{L} n t\right)$$

Use (infinite) superposition!

$$x'' + \omega_0^2 x = 1 \longrightarrow x_p = \frac{1}{\omega_0^2}$$

$$x'' + \omega_0^2 x = \begin{cases} \cos\omega t & \begin{cases} x_p = \frac{1}{\omega_0^2-\omega^2}\cos\omega t & \omega \neq \omega_0 \\ x_p = \frac{1}{2}\frac{t}{\omega_0}\sin\omega_0 t & \omega = \omega_0 \end{cases} \\ \sin\omega t & \begin{cases} x_p = \frac{1}{\omega_0^2-\omega^2}\sin\omega t & \omega \neq \omega_0 \\ x_p = -\frac{t}{2\omega_0}\cos\omega_0 t & \omega = \omega_0 \end{cases} \end{cases}$$

So, soln to $x''(t) + \omega_0^2 x(t) = f(t)$

is
$$x(t) = \frac{a_0}{\omega_0^2} + \sum_{n=1}^{\infty} a_n \frac{1}{\omega_0^2 - (\frac{\pi}{L}n)^2}\cos\left(\tfrac{\pi}{L}nt\right) + \sum_{n=1}^{\infty} b_n \frac{1}{(\omega_0^2 - (\frac{\pi}{L}n)^2)}\sin\left(\tfrac{\pi}{L}nt\right)$$

provided $\nexists\, n$ s.t. $\frac{\pi}{L} n = \omega_0$ i.e. $\frac{2\pi}{\omega_0} = \frac{2L}{n} = \frac{\text{forcing period}}{n}$

If the natural period is an integer fraction $\frac{1}{n}\cdot$ ~~forcing~~ forcing period, then there is resonance iff one of the corresponding a_n, or $b_n \neq 0$!!