Math 2250-1
Friday Dec. 5

§9.1-9.2 Fourier series

HW for Friday Dec. 12
will include

9.3 1, 9, 17, 19, 20
9.4 1, 7, 9, 13

Recall from Wed.

$f: \mathbb{R} \to \mathbb{R}$ 2π-periodic

Has Fourier series

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt = \langle f, \tfrac{1}{\sqrt{2}} \rangle \tfrac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} \langle f, \cos nt \rangle \cos nt + \sum_{n=1}^{\infty} \langle f, \sin nt \rangle \sin nt$$

So
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

where $\langle f, g \rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt$

and $\{ \frac{1}{\sqrt{2}}, \cos t, \cos 2t, \ldots, \sin t, \sin 2t, \ldots \}$
are orthonormal.

- Finish Example 2, p. 4 Wed. ("sawtooth fcn")
- Review Theorems page 5 Wed (1 = distance conv. thm
  2 = pointwise conv. thm
  3 = differentiation thm)
- Do Example 3 p. 5 (squarewave)

<u>Remark</u>: We can actually prove the differentiation theorem:

if $f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$

and if

$f' \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nt + \sum_{n=1}^{\infty} B_n \sin nt$

then

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\,dt = \frac{1}{\pi}\left[f(\pi) - f(-\pi) \right] = 0 \quad \text{(since } f \text{ cont. \& } 2\pi\text{-peri)}$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\cos nt\,dt = \frac{1}{\pi}\left[f(t)\cos nt \Big|_{-\pi}^{\pi} - -n\int_{-\pi}^{\pi} f(t)\sin nt\,dt \right] = n b_n$$

$du = f'(t)dt \quad v = \cos nt$
$u = f(t) \quad dv = -n \sin nt\, dt$

and by same reasoning, $B_n = -n a_n$ ∎

Remark Because $\{\frac{1}{\sqrt{2}}, \cos t, \cos 2t, \dots \cos nt, \sin t, \sin 2t, \dots \sin nt\}$ are an orthonormal basis for their span V_n,

if $f(t) = \alpha_0 + \sum_{k=1}^{\cancel{\infty} n} \alpha_k \cos kt + \sum_{k=1}^{n} \beta_k \sin kt \in V_n$

then it must be that

$$\left.\begin{aligned} \alpha_0 &= \tfrac{a_0}{2} \\ \alpha_k &= a_k \\ \beta_k &= b_k \end{aligned}\right\} \text{ the Fourier coeffs.}$$

pf: e.g. $a_\ell = \langle f, \cos \ell t\rangle = \langle \alpha_0 + \sum_{k=1}^{n} \alpha_k \cos kt + \sum \beta_k \sin kt, \cos \ell t\rangle$

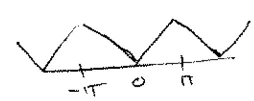
$= 0 + \alpha_\ell + 0$

Example 4 What is the 2π-periodic Fourier series for

a) $\sin 5t - 8\cos 10t$?

b) $\cos^2 3t$?

Example 5 (magic formulas – more in Hw). We showed for "tent fcn" $f(t) = |t| \quad -\pi \le t \le \pi$



$$f(t) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos nt}{n^2}$$

by Conv. thm. $f(t) =$ its Fourier series

a) Compute $f(\pi)$ via Fourier to deduce $\sum_{n \text{ odd}} \frac{1}{n^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \dots = \frac{\pi^2}{8}$

b) Use (a) and cleverness to deduce $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots = \frac{\pi^2}{6}$

So far, we've assumed $f(t)$ is 2π-periodic.

What if $g(u)$ is $2L$-periodic?

then the corresponding inner product $\langle g, h\rangle := \frac{1}{L}\int_{-L}^{L} g(u)h(u)\,du$

makes $\left\{ \frac{1}{\sqrt{2}}, \cos\frac{\pi}{L}u, \cos\frac{2\pi}{L}u, \dots \cos\frac{k\pi}{L}u, \sin\frac{\pi}{L}u, \sin\frac{2\pi}{L}u, \dots \sin\frac{k\pi}{L}u, \dots \right\}$

orthonormal!

(proof: substitute $\frac{\pi}{L}u = t$; $u = \frac{L}{\pi}t$ to reduce to previous computations)

so the Fourier series for g is defined by

$$g \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}u\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}u\right)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} g(u)\,du$$

$$a_n = \frac{1}{L}\int_{-L}^{L} g(u)\cos(nu)\,du$$

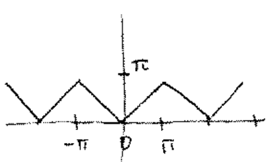
$$b_n = \frac{1}{L}\int_{-L}^{L} g(u)\sin(nu)\,du$$

(and then we use "t" instead of "u").

You can also just rescale Fourier series to get $2L$-periodic ones from 2π-periodic ones

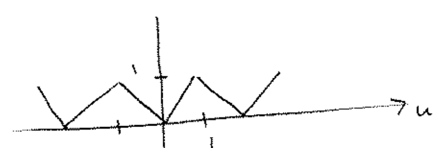
Example 6

did



$$f \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos nt}{n^2}$$

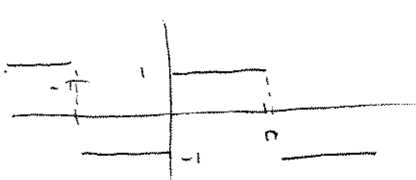
so,



$g(u) = \frac{1}{\pi}f(\pi u)$ so

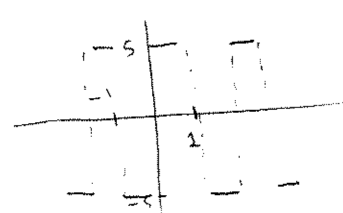
$g \sim ?$

did



$$f \sim \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n^2}$$

so



$g(u) = 5f(\pi u)$

so $g \sim$