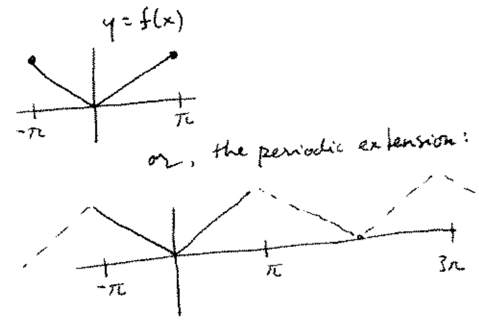
Math 2280-1
Wednesday Dec. 3

Chapter 9: Fourier series & applications to DE's and PDE's
$\uparrow$ partial differential equations

Recall (?) from 2270:



Let $f : [-\pi, \pi] \to \mathbb{R}$ be piecewise continuous

(or equivalently, extend $f : \mathbb{R} \to \mathbb{R}$ as a 2π-periodic, p.w. cont. function)

Fourier coefficients of f:

$$a_0 := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \qquad n \in \mathbb{N}$$

$$b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \qquad n \in \mathbb{N}$$

$$\boxed{\text{Fourier series for } f: \quad f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt}$$

Motivation (2270 "orthogonality" chapter)

$V := \{ f : \mathbb{R} \to \mathbb{R},\ f \text{ is } 2\pi \text{ periodic } (f(x+2\pi) = f(x)\ \forall x), \text{ and p.w. continuous} \}$

inner product

$$\langle f, g \rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt \quad \text{satisfies the axioms} \quad \begin{cases} a)\ \langle f,f\rangle \geq 0,\ \langle f,f\rangle = 0 \text{ iff } f = 0 \\ b)\ \langle f,g\rangle = \langle g,f\rangle \\ c)\ \langle f, g+h\rangle = \langle f,g\rangle + \langle f,h\rangle \\ d)\ \langle sf, g\rangle = \langle f, sg\rangle = s\langle f,g\rangle \end{cases} \quad \forall f,g \in V,\ s \in \mathbb{R}$$

So, $\|f\| = \langle f,f\rangle^{1/2}$, $f \perp g$ iff $\langle f,g\rangle = 0$, $\measuredangle f,g = \cos^{-1}\left(\frac{\langle f,g\rangle}{\|f\|\,\|g\|}\right)$, projection, etc!

$\mathrm{dist}(f,g) = \|f-g\|$,

In particular, if $\{u_1, u_2, \ldots u_k\}$ are orthonormal and span a subspace W of V

then

$$\mathrm{proj}_W f = \langle f,u_1\rangle u_1 + \langle f,u_2\rangle u_2 + \cdots + \langle f,u_k\rangle u_k \quad \ldots$$

is the nearest function in W, to f.

(just like you proved for dot product, using only (a)-(d)!)

<u>Theorem</u> Let $V_N = \text{span}\left\{ \frac{1}{\sqrt{2}}, \cos t, \cos 2t, \ldots \cos Nt, \sin t, \sin 2t, \ldots \sin Nt \right\}$

The $2N+1$ functions in this list are orthonormal!

Thus for any $f \in V$,

$$\text{proj}_{V_N} f = \underbrace{\langle f, \tfrac{1}{\sqrt{2}} \rangle \tfrac{1}{\sqrt{2}}}_{\frac{a_0}{2}} + \sum_{n=1}^{N} \underbrace{\langle f, \cos nt \rangle}_{a_n} \cos nt + \sum_{n=1}^{N} \underbrace{\langle f, \sin nt \rangle}_{b_n} \sin nt$$

i.e., the truncated Fourier series

(the idea is that as $N \to \infty$, the $\text{proj}_{V_N} f$ approximate f effectively; see later theorems)

proof:

Check that $\left\{ \frac{1}{\sqrt{2}}, \cos t, \cos 2t, \ldots \cos nt, \sin t, \sin 2t, \ldots \sin nt \right\}$ are orthonormal:

Hint:

$$\cos(m+k)t = \cos mt \cos kt - \sin mt \sin kt$$
$$\sin(m+k)t = \cos mt \sin kt + \sin mt \cos kt$$

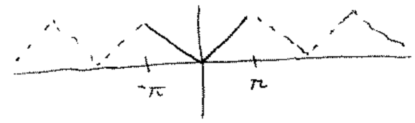
$\Longrightarrow$

$$\cos mt \cos kt = \tfrac{1}{2}\left[\cos(m+k)t + \cos(m-k)t\right] \quad \text{(even if } m=k!)$$
$$\sin mt \sin kt = \tfrac{1}{2}\left[\cos(m-k)t - \cos(m+k)t\right] \quad \text{(even if } m=k!)$$
$$\cos mt \sin kt = \tfrac{1}{2}\left[\sin(m+k)t + \sin(-m+k)t\right]$$

Example 1 : an even function $(f(-x) = f(x))$

$f(t) = |t|$ for $-\pi \le t \le \pi$

extended to be 2π-periodic :



find a_0, a_n, b_n

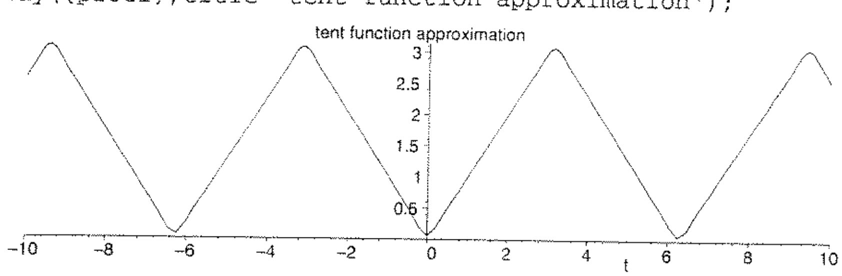
ans :

$$f \sim \frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{1}{9}\cos 3t + \frac{1}{5^2}\cos 5t + \dots\right)$$

```
> f9:=t->Pi/2-4/Pi*sum(cos((2*j+1)*t)/(2*j+1)^2,j=0..4);
```

$$f9 := t \to \frac{\pi}{2} - \frac{4\left(\displaystyle\sum_{j=0}^{4} \frac{\cos((2j+1)t)}{(2j+1)^2}\right)}{\pi}$$

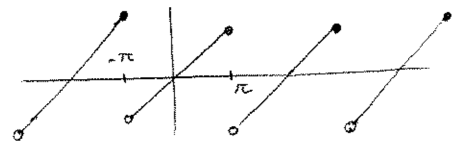
```
> plot1:=plot(f9(t),t=-10..10,color=black):
  display({plot1},title='tent function approximation');
```



Example 2 an odd fcn ($f(-x) = -f(x)$)

$f(t) = t$ for $-\pi < t \le \pi$

extended to be 2π-periodic



find a_0, a_n, b_n

ans :

$$f \sim 2\left(\sin t - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \dots\right.$$

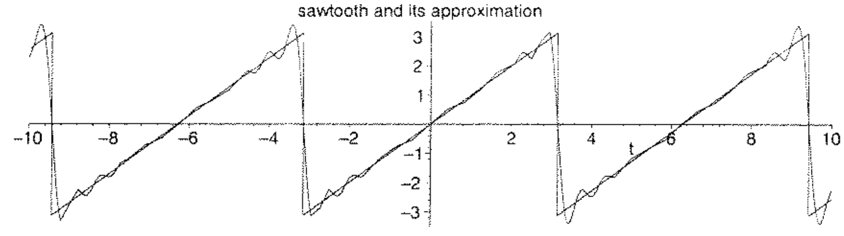
$$= 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nt}{n}$$

Can you deduce anything about Fourier coeffs of 2π-periodic even functions? of 2π-periodic odd functions?

```
> g10:=t->2*sum((-1)^(n+1)*sin(n*t)/n,n=1..10);
```

$$g10 := t \to 2\left(\sum_{n=1}^{10} \frac{(-1)^{(n+1)}\sin(n\,t)}{n}\right)$$

```
> plot3:=plot(g10(t),t=-10..10,color=black):
  plot4:=plot(t-2*Pi*Heaviside(t-Pi)-2*Pi*Heaviside(t-3*Pi)-
  2*Pi*(Heaviside(t+Pi)-1)-2*Pi*(Heaviside(t+3*Pi)-1),t=-10..10,colo
  r=black):
  display({plot3,plot4},title='sawtooth and its approximation');
```



Convergence theorems: (Need more analysis than you know yet to prove these)

1. Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic and piecewise continuous.

   Let $f_N = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos nt + \sum_{n=1}^{N} b_n \sin nt$

$$\text{Then } \lim_{N\to\infty} \| f - f_N \| = \lim_{N\to\infty} \left[\frac{1}{\pi} \int_{-\pi}^{\pi} (f(t) - f_N(t))^2 \, dt \right]^{1/2} = 0$$

the way we measure "distance" between functions for our given inner product

2. If f is as above and piecewise differentiable, with jump discontinuities, then for any

   t_0 s.t. f is differentiable at t_0, $\lim_{N\to\infty} f_N(t_0) = f(t_0)$

   for any

   t_0 s.t. f has a jump at t_0, from $f_-(t_0) := \lim_{t \nearrow t_0^-} f(t)$

$$f_+(t_0) = \lim_{t \searrow t_0^+} f(t)$$

   then $\lim_{N\to\infty} f_N(t_0) = \frac{1}{2}\left(f_-(t_0) + f_+(t_0)\right)$, i.e. the average of the left & right limits

   [you can see this illustrated in the sawtooth example].

3. If f is continuous & piecewise diff'ble, and f' is piecewise continuous, and if

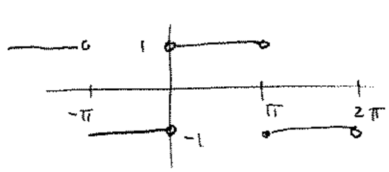
$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

$$\text{then } f' \sim \sum_{n=1}^{\infty} -n a_n \sin nt + \sum_{n=1}^{\infty} n b_n \cos nt$$

   i.e. you can differentiate (& antidiff) term by term.

Example 3. Use Example 1 to find the square-wave Fourier series

$$sq(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & -\pi < t < 0. \end{cases}$$



ans: $sq \sim \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n}$