Math 2280-1
Tuesday December 2

§ 7.4-7.5 Bonus day
(you only need to do HW & be tested on 7.1-7.3)

Today we'll talk about
the translation theorem (2b),
the convolution theorem (3),

with applications which are "new"!!

$f(t)$	$F(s)$	
$f(t)$	$\int_0^\infty e^{-st} f(t)\,dt$	
$c_1 f(t) + c_2 g(t)$	$c_1 F(s) + c_2 G(s)$	
1	$1/s$	
t	$1/s^2$	
t^n	$n!/s^{n+1}$	$n \in \mathbb{N}$
e^{at}	$1/(s-a)$	
$\cos kt$	$s/(s^2+k^2)$	
$\sin kt$	$k/(s^2+k^2)$	
$t^n e^{at}$	$n!/(s-a)^{n+1}$	
$e^{at}\cos kt$	$(s-a)/((s-a)^2+k^2)$	
$e^{at}\sin kt$	$k/((s-a)^2+k^2)$	
$t\cos kt$	$(s^2-k^2)/(s^2+k^2)^2$	
$\frac{t}{2k}\sin kt$	$\frac{s}{(s^2+k^2)^2}$	
$\frac{1}{2k^3}(\sin kt - kt\cos kt)$	$\frac{1}{(s^2+k^2)^2}$	
(1a) $f'(t)$	$sF(s) - f(0)$	
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	
$f'''(t)$	$s^3F(s) - s^2f(0) - sf'(0) - f''(0)$	
	etc.	
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$	
(1b) $tf(t)$	$-F'(s)$	
$t^2 f(t)$	$F''(s)$	
$t^n f(t)$	$(-D_s)^n F(s)$	
$f(t)/t$	$\int_s^\infty F(\sigma)\,d\sigma$	
(2a) $e^{at}f(t)$	$F(s-a)$	
(2b) $u(t-a)f(t-a)$	$e^{-as}F(s)$	
(3) $(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	

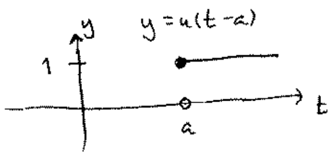
we got this far Monday.

(2b) The unit step function

$$u(t) := \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

so $u(t-a) = \begin{cases} 1, & t-a \geq 0 \quad (t \geq a) \\ 0 & t < a \end{cases}$

turn forcing functions on and off!



$$\mathcal{L}\{u(t-a)\}(s) = \int_0^\infty e^{-st} u(t-a)\,dt$$

$$= \int_0^a 0 + \underbrace{\int_a^\infty e^{-st}\,dt}_{\left.\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s} \quad (s>0)}$$

$\mathcal{L}\{u(t-a)f(t-a)\}(s)$

$$= \int_0^\infty \text{—}\,dt = \int_a^\infty e^{-st} f(t-a)\,dt \qquad \tilde{t} = t-a$$

$$= \int_0^\infty e^{-s(\tilde{t}+a)} f(\tilde{t})\,d\tilde{t}$$

$$= e^{-as} \underbrace{\int_0^\infty e^{-s\tilde{t}} f(\tilde{t})\,d\tilde{t}}_{F(s)} \quad \blacksquare$$

Example (p 484)

$m = 32$ lb ($m = 1$ slug), $c = 0$,

$k = 4$ lb/ft. mass initially at equilibrium $(x(0)=0, x'(0)=0)$

at $t=0$ apply $f(t) = \cos 2t$

at $t = 2\pi$ force is turned off. Solve IVP $\begin{cases} x'' + 4x = f(t) \\ x(0)=0 \\ x'(0)=0 \end{cases}$

soln

$$s^2 X(s) - 0 - 0 + 4X(s) = F(s)$$

$$X(s)(s^2+4) = F(s)$$

$$f(t) = \cos 2t\,(1 - u(t-2\pi))$$
$$= \cos 2t - \cos 2t\, u(t-2\pi)$$
$$= \cos 2t - \cos(2(t-2\pi))\, u(t-2\pi)$$

$$F(s) = \frac{s}{s^2+4} - e^{-2\pi s}\frac{s}{s^2+4}$$

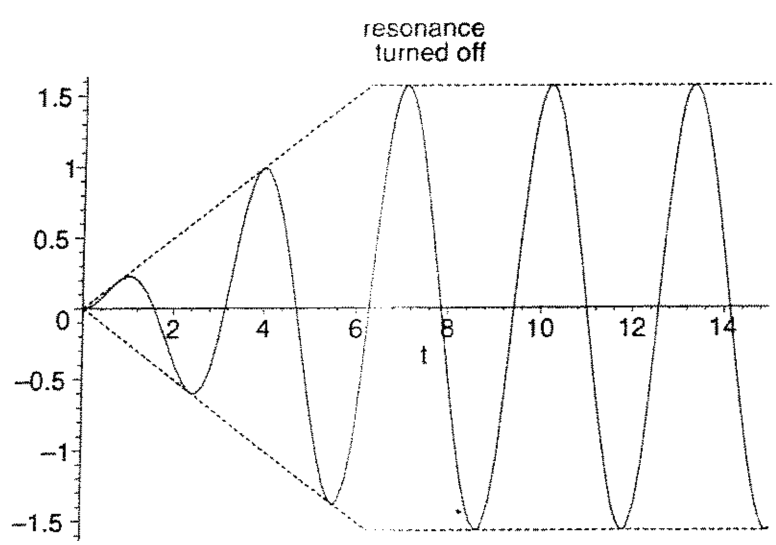
$$X(s) = \frac{1}{s^2+4}\left(\frac{s}{s^2+4} - e^{-2\pi s}\frac{s}{s^2+4}\right)$$
$$= \frac{s}{(s^2+4)^2} - e^{-2\pi s}\frac{s}{(s^2+4)^2}$$

Table!

$$x(t) = \frac{1}{4}t\sin 2t - u(t-2\pi)\frac{1}{4}(t-2\pi)\sin 2t$$

$$= \begin{cases} \frac{t}{4}\sin 2t & 0 \le t < 2\pi \\ \frac{\pi}{2}\sin 2t & t \geqslant 2\pi \end{cases}$$

```
> with(plots):
> plot1:=plot(.25*t*sin(2*t)-Heaviside(t-2*Pi)*(t-2*Pi)/4*sin(2*t),
       t=0..15,color=black):
  plot2:=plot(t/4,t=0..2*Pi,color=black,linestyle=2):
  plot3:=plot(-t/4,t=0..2*Pi,color=black,linestyle=2):
  plot4:=plot(Pi/2,t=2*Pi..15,color=black,linestyle=2):
  plot5:=plot(-Pi/2,t=2*Pi..15,color=black,linestyle=2):
  display({plot1,plot2,plot3,plot4,plot5},title=`resonance
    turned off`);
```



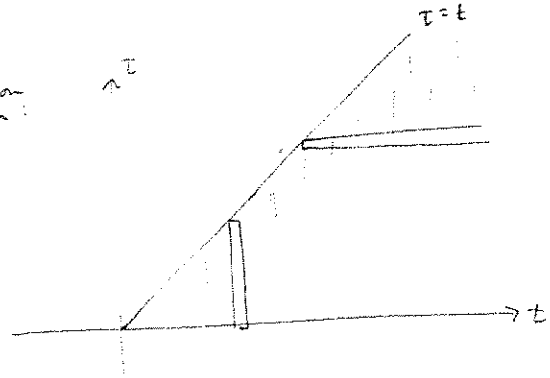
~~①~~ yesterday
③ today

proof of convolution theorem:
(is a good review of iterated integrals)

$$\mathcal{L}\{f*g\}(s) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(t-\tau)d\tau\right) dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(\tau) g(t-\tau)\, d\tau\, dt$$

integration region:



interchange limits:

$$= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\, g(t-\tau)\, dt\, d\tau$$

$$= \int_0^\infty \int_\tau^\infty e^{-s\tau} f(\tau)\, e^{-s(t-\tau)} g(t-\tau)\, dt\, d\tau \qquad \text{(pattern recognition)}$$

$$= \int_0^\infty e^{-s\tau} f(\tau) \left[\int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\, dt \right] d\tau$$

$\tilde{t} = t - \tau$
$d\tilde{t} = dt$

$$\underbrace{\left[\int_0^\infty e^{-s\tilde{t}} g(\tilde{t})\, d\tilde{t} \right]}_{G(s)}$$

$$= G(s) \int_0^\infty e^{-s\tau} f(\tau)\, d\tau$$

$$= G(s)\, F(s) \qquad !!$$

__example__: verify the theorem
for $f(t) = \sin t$
$g(t) = \cos t$
(you may need trig id $\sin^2 \tau = \dfrac{1 - \cos 2\tau}{2}$)

__Remark:__
$$f*g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$
$$= g*f(t)$$
(by substituting $\tilde{\tau} = t - \tau$)

Application: for any nonhomogeneous, const. coeff linear DE or system, there is a convolution formula for the solution (must be equivalent to variation of parameters ?!)

example

$$\begin{cases} mx'' + cx' + kx = f(t) \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

$\mathcal{L}$: $X(s)\left[ms^2 + cs + k\right] = F(s) + (sx_0 + v_0)m + x_0 c$

$$X(s) = \underbrace{\frac{F(s)}{ms^2+cs+k}}_{\text{use convolution formula to invert}} + \underbrace{\frac{(sx_0+v_0)m + x_0 c}{ms^2+cs+k}}_{\text{table}}$$

subexample

$$\begin{cases} x'' + x = f(t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

$\mathcal{L}$: $X(s)(s^2+1) = F(s)$

$$X(s) = \frac{1}{s^2+1} F(s)$$

$$x(t) = (\sin * f)(t) = \int_0^t f(\tau)\sin(t-\tau)\,d\tau = \int_0^t (\sin\tau) f(t-\tau)\,d\tau$$

see handout & play the resonance game.

Math 2280-1
December 2, 2008
Guess the resonance game, using convolution formula, section 7.4

```
> with(plots):with(inttrans):
    #the library inttrans includes Laplace
```

We are considering the undamped forced harmonic oscillator

$$x''(t) + x(t) = f(t)$$

with initial data x(0)=v(0)=0. When we take the Laplace transform of this equation we deduce

$$X(s) = \frac{1\,F(s)}{s^2+1}$$

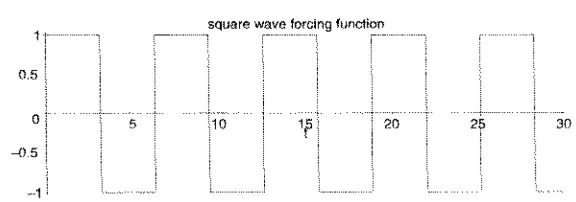
so that the convolution theorem implies $x(t) = \sin * f(t)$. Since the unforced system has a natural angular frequency $\omega_0 = 1$, we expect resonance when the forcing function has the corresponding period of $\frac{2\pi}{w_0} = 2\pi$. We will discover that there is a surprising possible error in our reasoning.

Example 1: A square wave forcing function with amplitude 1 and period 2π. Let's talk about how we came up with the formula (which works until $t = 11\pi$).

```
> f:=t->-1+2*sum((-1)^n*Heaviside(t-n*Pi),n=0..10);
   #Heaviside was an early user of the unit step function
   #and so Maple names it after him
```

$$f := t \to -1 + 2\left(\sum_{n=0}^{10} (-1)^n \,\mathrm{Heaviside}(t - n\pi)\right)$$

```
> plot(f(t),t=0..30,color=black,title='square wave forcing
  function');
```




```
> x:=t->int(sin(t-tau)*f(tau),tau=0..t);
     #convolution formula for the solution
```

$$x := t \to \int_0^t \sin(t-\tau)\, f(\tau)\, d\tau$$

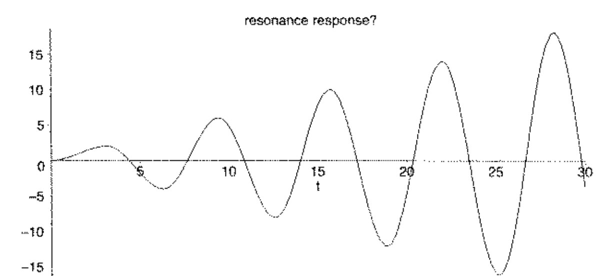
You actually could use the convolution formula to work out x(t) by hand, if you wished to do so!

```
> x(t);   #you actually could work this out by hand!
```

$$-1 - 2\,\mathrm{Heaviside}(t-3\pi) + 2\,\mathrm{Heaviside}(t) - 2\,\mathrm{Heaviside}(t-3\pi)\cos(t) - 2\,\mathrm{Heaviside}(t)\cos(t)$$
$$+ 2\,\mathrm{Heaviside}(t-4\pi) - 2\,\mathrm{Heaviside}(t-\pi) - 2\,\mathrm{Heaviside}(t-4\pi)\cos(t) - 2\,\mathrm{Heaviside}(t-\pi)\cos(t)$$
$$- 2\,\mathrm{Heaviside}(t-5\pi) + 2\,\mathrm{Heaviside}(t-2\pi) - 2\,\mathrm{Heaviside}(t-5\pi)\cos(t)$$
$$- 2\,\mathrm{Heaviside}(t-2\pi)\cos(t) + 2\,\mathrm{Heaviside}(t-6\pi) - 2\,\mathrm{Heaviside}(t-9\pi)\cos(t)$$
$$- 2\,\mathrm{Heaviside}(t-6\pi)\cos(t) + 2\,\mathrm{Heaviside}(t-10\pi) - 2\,\mathrm{Heaviside}(t-7\pi)$$
$$- 2\,\mathrm{Heaviside}(t-10\pi)\cos(t) - 2\,\mathrm{Heaviside}(t-7\pi)\cos(t) + 2\,\mathrm{Heaviside}(t-8\pi)$$
$$- 2\,\mathrm{Heaviside}(t-8\pi)\cos(t) - 2\,\mathrm{Heaviside}(t-9\pi) + \cos(t)$$

As expected (?), we get resonance.

```
> plot(x(t),t=0..30,color=black,title='resonance response?');
```

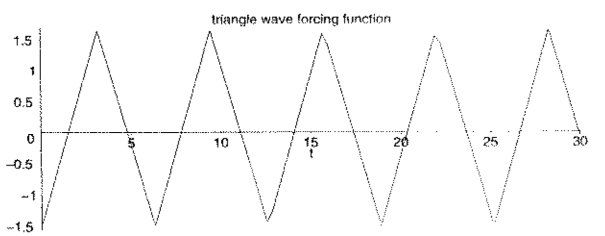


Example 2: triangle wave forcing function, same period.

```
> g:=t->int(f(u),u=0..t)-1.5;
  #this should be a triangle wave...
```

$$g := t \to \int_0^t f(u)\,du - 1.5$$

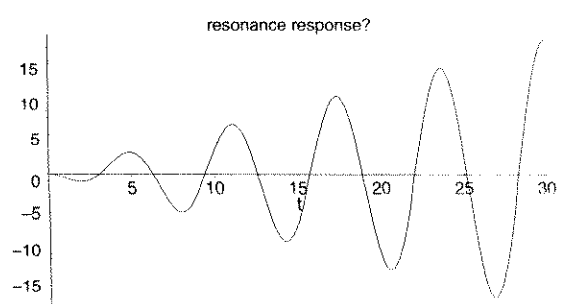
```
> plot(g(t),t=0..30,color=black, title='triangle wave forcing
  function');
```




```
> y:=t->int(sin(t-tau)*g(tau),tau=0..t);
```

$$y := t \to \int_0^t \sin(t-\tau)\,g(\tau)\,d\tau$$

```
> plot(y(t),t=0..30,color=black, title='resonance response?');
```

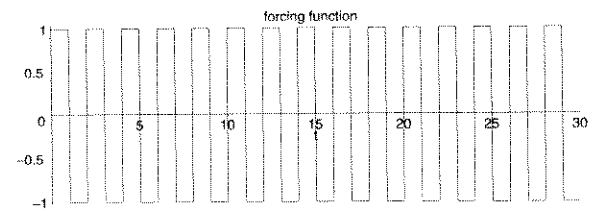


Example 3: Now let's force with a period which is not the natural one. This square wave has period 2.

```
> h:=t->-1+2*sum((-1)^n*Heaviside(t-n),n=0..30);
```

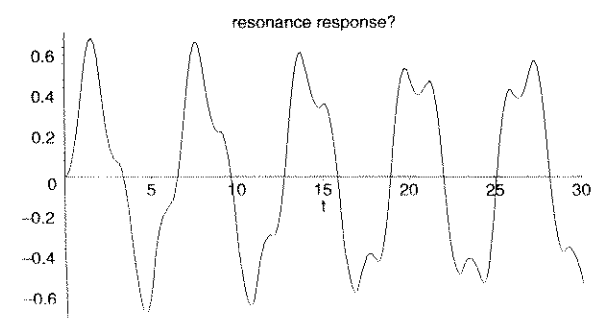
$$h := t \to -1 + 2\left(\sum_{n=0}^{30} (-1)^n \operatorname{Heaviside}(t-n)\right)$$

```
> plot(h(t),t=0..30,color=black, title='forcing function');
```




```
> z:=t->int(sin(t-tau)*h(tau),tau=0..t);
  plot(z(t),t=0..30,color=black,title='resonance response?');
```

$$z := t \to \int_0^t \sin(t-\tau)\,h(\tau)\,d\tau$$

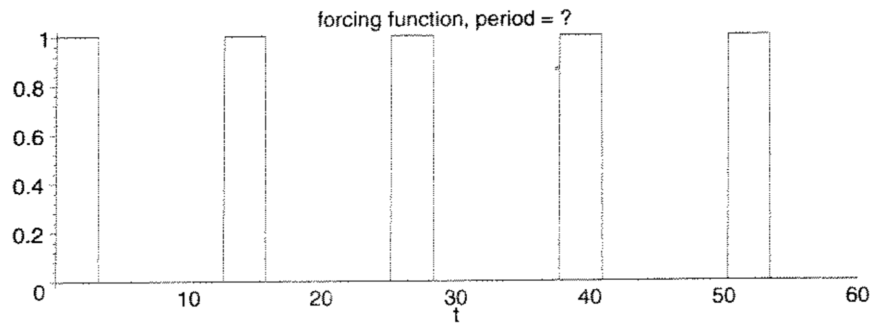


Example 4: A square wave which does not have the natural period, so we don't expect resonance?

```
> k:=t->sum(Heaviside(t-4*Pi*n)-Heaviside(t-4*Pi*n-Pi),
    n=0..5);
```

$$k := t \to \sum_{n=0}^{5} (\text{Heaviside}(t - 4\,n\,\pi) - \text{Heaviside}(t - 4\,n\,\pi - \pi))$$

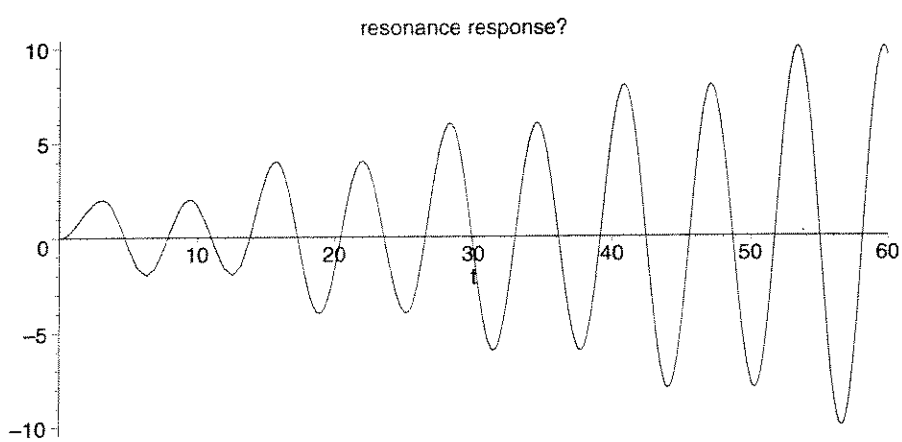
```
> plot(k(t),t=0..60,color=black,title=`forcing function, period =
  ?`);
>
```




```
> w:=t->int(sin(t-tau)*k(tau),tau=0..t);
```

$$w := t \to \int_0^t \sin(t - \tau)\,k(\tau)\,d\tau$$

```
> plot(w(t),t=0..60,color=black,title=`resonance response?`);
```



Hey, what happened???? How do we need to modify our thinking if we force a system with something which is not sinusoidal, in terms of worrying about resonance?