Math 2280-1
Wednesday December 10
Bonus Day!
Survey of §9.5–9.6 : heat and wave partial differential equations

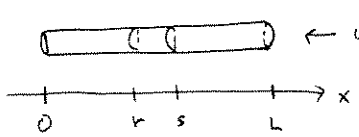
9.5 1 space dimension heat equation for temperature $u(x,t)$ ← time $t \geq 0$

x: location $0 \leq x \leq L$

$$\boxed{u_t = k\, u_{xx}}$$

(or $u_t = k u_{xx} + f(x,t)$ for non-homogeneous heat eqtn)
"k" is called <u>thermal diffusivity</u>, depends on material, see table p. 622

<u>derivation</u>



← cross-sectional area a

"Heat" is a measurement of energy
If $u(x,t)$ is absolute temp (e.g. °Kelvin)
then the energy is test interval
$r \leq x \leq s$ is

$$E(t) = \int_r^s u(x,t)\, c\, \delta\, \underbrace{a\, dx}_{dV \ \text{cm}^3}$$

δ: density gm/cm³; $\underbrace{\delta\, a\, dx}_{dm}$ mass gm

c: specific heat calories/gm
= energy required to heat 1 gm by 1° C,

$c\,\delta\, a\, dx$ = energy required to heat dx-piece 1°.

$u(x,t)\, c\,\delta\, a\, dx$ = energy in dx-piece.

Assuming lateral insulation
(so heat can only escape through ends),
assume

$$\frac{dE}{dt} = K a \left(u_x(s) - u_x(r) \right) \qquad \text{Heat flux proportional to temperature gradient}$$

$$\int_r^s u_t\, c\delta a\, dx = K a \int_r^s u_{xx}\, dx \quad (\text{FTC})$$

$$\lim_{s \to r} \frac{1}{s-r} : \quad \Rightarrow \quad u_t\, c\delta a = K a\, u_{xx}$$

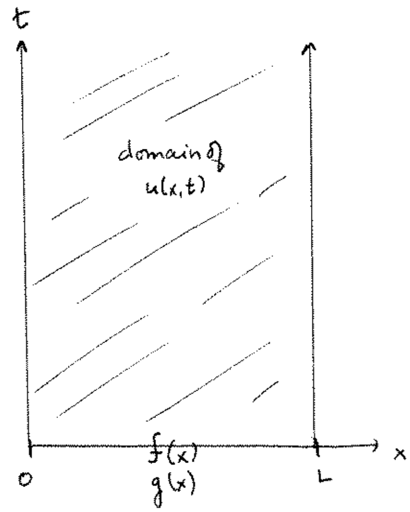
$$\boxed{u_t = k u_{xx}, \qquad k = \frac{K}{c\delta}}$$

- If u satisfies $u_t = ku_{xx}$ then so does $v = c_1 u + c_2$, so you may use Celsius, Farenheit, Kelvin
- $L(u) = u_t - ku_{xx}$ is linear so principle of superposition holds.

Two of the most-usually studied IBVP's (Initial boundary value problems)

1 $\begin{cases} u_t = ku_{xx} & 0<t<\infty \\ & 0<x<L \\ u(x,0) = f(x) & \text{init. temp.}\ 0<x<L \\ u(0,t) = 0 & t>0 \\ u(L,t) = 0 & t>0 \end{cases}$

boundary temp held const. (need not always be zero).



2 $\begin{cases} u_t = ku_{xx} & 0<t<\infty \\ & 0<x<L \\ u(x,0) = g(x) & 0<x<L \\ u_x(0,t) = 0 & t>0 \\ u_x(L,t) = 0 & t>0 \end{cases}$

no heat flux thru boundary (insulated end condition)

Example 1 $\boxed{f(x) = \sin\left(\frac{\pi}{L}x\right)}$

product sol'n!

$$u(x,t) = v(t)\sin(\omega x), \quad v(0) = 1$$
$$u_t = v'(t)\sin\omega x$$
$$u_{xx} = v(t)(-\omega^2)\sin\omega x$$
$$\Leftrightarrow \begin{cases} v' = -\omega^2 v \\ v(0) = 1 \end{cases} \Leftrightarrow v(t) = e^{-\omega^2 t}$$

so $\boxed{u(x,t) = \sin\left(\frac{\pi}{L}x\right) e^{-\left(\frac{\pi}{L}\right)^2 t}}$

solves 1!

general sol'n, use sine series for f & superpose!

$$f \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L}nx\right)$$
$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L}nx\right) e^{-\left(\frac{\pi}{L}n\right)^2 t}$$

2 $g(x) = \cos\left(\frac{\pi}{L}x\right)$

$\boxed{u(x,t) = \cos\left(\frac{\pi}{L}x\right) e^{-\left(\frac{\pi}{L}\right)^2 t}}$!

(see LHS)

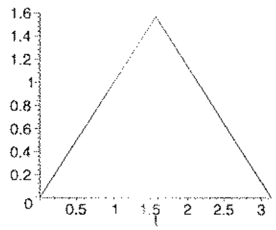
general sol'n, use cosine series for $g(x)$ & superpose product sol'ns!

$$g \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L}nx\right)$$
$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L}nx\right) e^{-\left(\frac{\pi}{L}n\right)^2 t}$$

(this method is called "separation of variables")

pictures next page, with tent function initial temperature

```
> tent:=t->(1-Heaviside(t-Pi/2))*t +Heaviside(t-Pi/2)*(Pi-t):
> plot(tent(t),t=0..Pi,color=black);
```



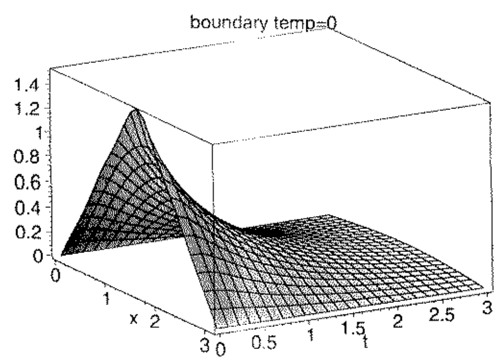
```
> sineseries:=proc(ff,L,b)   #ff=function, L=half period
  local m,  #dummy letter to index coefficients
        s;  #domain variable
  assume(m,integer):
  b:=m->simplify(2/L*int(ff(s)*sin(Pi/L*m*s),s=0..L)):
  end:
> sineseries(tent,Pi,B):
> B(n);
```

$$\frac{4\sin\left(\frac{\pi n\sim}{2}\right)}{\pi n\sim^2}$$

```
> u1:=(x,t)->sum(B(n)*sin(n*x)*exp(-n^2*t),n=1..10);
```

$$u1 := (x,t) \to \sum_{n=1}^{10} B(n)\sin(n\,x)\,e^{(-n^2 t)}$$

```
> plot3d(u1(x,t),x=0..Pi,t=0..3,axes=boxed,title='boundary temp=0');
```



$$\begin{cases} u_t = u_{xx} \\ u(x,0) = tent(x) \\ u(0,t) = u(\pi,t) = 0 \quad t>0 \end{cases}$$

fixed boundary temp.

$\lim_{t\to\infty} u(x,t) = ?$

```
> cosseries:=proc(ff,L,a)   #ff=function, L=half period
  local m,  #dummy letter to index coefficients
        s;  #domain variable
  assume(m,integer):
  a:=m->simplify(2/L*int(ff(s)*cos(Pi/L*m*s),s=0..L)):
  end:
> cosseries(tent,Pi,A):
> A(0);
  A(n);
```

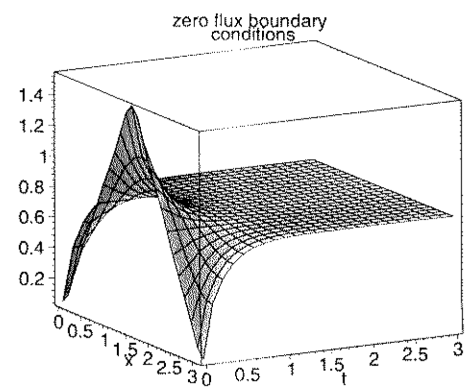
$$\frac{\pi}{2}$$

$$-\frac{2\left((-1)^{n\sim} - 2\cos\left(\frac{\pi n\sim}{2}\right) + 1\right)}{\pi n\sim^2}$$

```
> u2:=(x,t)->Pi/4+sum(A(n)*cos(n*x)*exp(-n^2*t),n=1..10);
```

$$u2 := (x,t) \to \frac{\pi}{4} + \left(\sum_{n=1}^{10} A(n)\cos(n\,x)\,e^{(-n^2 t)}\right)$$

```
> plot3d(u2(x,t),x=0..Pi,t=0..3,axes=boxed,title='zero flux boundary
  conditions');
```

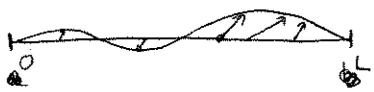


$$\begin{cases} u_t = u_{xx} \\ u(x,0) = tent(x) \\ u_x(0,t) = u_x(\pi,t) = 0 \end{cases}$$

no heat flux

$\lim_{t\to\infty} u(x,t) = ?$

§9.6
Vibrating strings & the wave equation



vibrating string.

We assume tension in string changes as it is stretched,

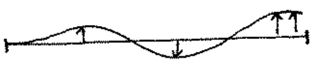
i.e. $T = \mathcal{T}(\rho)$ $\quad \rho$ = density $\quad$ (probably $\mathcal{T}(\rho) \searrow$ for ρ)

$$T = \underbrace{\mathcal{T}(\rho_0)}_{T_0} + \mathcal{T}'(\rho_0)(\rho - \rho_0)$$

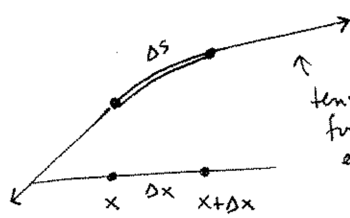
linearized model; T_0, ρ_0 for stationary (but stretched) string.

<u>case 1</u> transverse (vertical) vibrations

$y = y(x,t)$ = vertical displacement



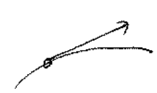
isolate a tiny piece, apply Newton's Law, linearize, to deduce wave eqtn:



tension force on ends

$$\underbrace{\rho \Delta s}_{\text{mass}} \; \underbrace{y_{tt}}_{\text{accel}} = \overbrace{\overset{\text{tension}}{T} \frac{y_x}{\sqrt{1+y_x^2}} \Bigg]_x^{x+\Delta x}}^{\text{net forces}}$$

vertical component of unit tang. vector to profile curve



$$\begin{bmatrix} 1 \\ y_x \end{bmatrix} \frac{1}{\sqrt{1+y_x^2}}$$

linearize

$$\frac{\Delta s}{\Delta x} \approx \sqrt{1+y_x^2} \approx 1$$

(y_x small $\Rightarrow y_x^2$ negligible)

$$\rho \Delta s = \rho_0 \Delta x$$

$$\Rightarrow \rho = \rho_0 \frac{ds}{dx} = \rho_0 \sqrt{1+y_x^2} \approx \rho_0$$

$$\Rightarrow T \approx T_0$$

$$\rho_0 (\Delta x) y_{tt} \approx T_0 \left(y_x(x+\Delta x) - y_x(x) \right)$$

$$\rho_0 y_{tt} \approx T_0 \left(\frac{y_x(x+\Delta x) - y_x(x)}{\Delta x} \right)$$

$$y_{tt} = \left(\frac{T_0}{\rho_0} \right) y_{xx}$$

$$\boxed{y_{tt} = a^2 y_{xx}}$$

$a = \sqrt{\frac{T_0}{\rho_0}}$ turns out to be wave speed.

<u>case 2</u> longitudinal motion, i.e. in direction of string, you get

$$x_{tt} = b^2 y_{xx}$$

where $b^2 = -T'(\rho_0)$, different speed!

What does "a" mean?

Special solution:

Let $f(z)$ a function of 1-variable.

consider $y(x,t) = f(x-at)$

$y_t = f'(x-at)(-a)$ chain rule!

$y_{tt} = f''(x-at)\, a^2$

$y_x = f'(x-at)\, 1$

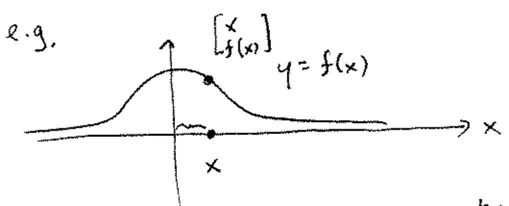
$y_{xx} = f''(x-at)\, 1$

$y_{tt} = a^2 y_{xx}$!

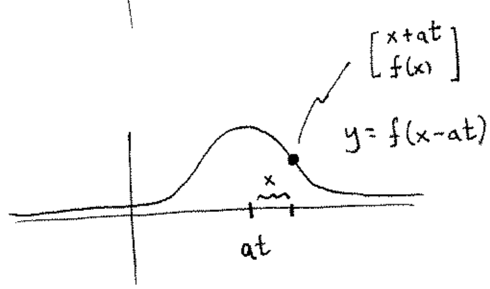
(also, $y(x,t) = f(x+at)$ solves the wave equation.)

$y(x,t) = f(x-at)$ is a wave (with constant "profile") moving to the right $\boxed{\text{with speed } a\,!}$

e.g.



profile at $t = 0$



profile at time t has moved over by amount at

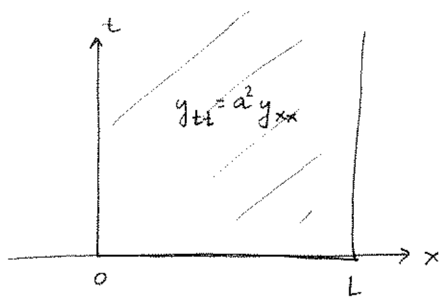
$y(x,t) = f(x+at)$ is a wave moving to the left!

(It will turn out that every solution to the wave eqtn $y_{tt} - a^2 y_{xx} = 0$ is a superposition of waves traveling to the left or right with speed a!)

In case 1, $a = \sqrt{\frac{T_0}{\rho_0}}$, so higher tension $\Rightarrow$ faster speed

lower density $\Rightarrow$ faster speed.

Natural IBVP's for wave eqtn:



$$\begin{cases} y_{tt}=a^2y_{xx} & t>0,\ 0<x<L \\ y(x,0)=f(x) & \text{initial displacement} \\ y_t(x,0)=g(x) & \text{initial velocity} \end{cases}$$

plus ① $y(0,t)=y(L,t)=0$ fixed endpoints

or ② $y_x(0,t)=y_x(L,t)=0$ free endpoints

Use Fourier series & superposition: $(2\times 2=4)$, and you can solve type 1 & type 2 problems!

① $\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{an\pi}{L}t\right)$

$y(x,0)=f(x)$ $\quad(y_t(x,0)=0)$
$y(0,t)=y(L,t)\equiv 0$

$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{an\pi}{L}t\right)$

$y_t(x,0)=g(x)$ $\quad(y(x,0)=0)$
$y(0,t)=y(L,t)=0$

② $\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{an\pi}{L}t\right)$

$y(x,0)=f(x)$ $\quad(y_t(x,0)=0)$
$y_x(0,t)=y_x(L,t)=0$

$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{an\pi}{L}t\right)$

$y_t(x,0)=g(x)$ $\quad(y(x,0)=0)$
$y_x(0,t)=y_x(L,t)=0$

It's possible I'll have a slinky and we can demo this!