Math 2280-1 Monday December 1

§7.1 - 7.3 More magic Laplace transform!

Fill in more of the table, then do some examples.

HW for Monday Dec. 8 (1)

7.1 (3) 7, (8, 13, 20) 21, (23, 28)
7.2 3, (4) 5, (6, 14) 19, (20, 28) 31
7.3 (3) 7, (8, 17, 20, 31)
7.4 2, 3, 36 (we'll discuss 7.4, but section is optional)
9.1 (6, 7, 10) 13, 15, 17, (20), (30)
9.2 (2, 9)

Finish (1a):

let $G(t) = \int_0^t f(\tau)\,d\tau$

$G'(t) = f(t)$
$G(0) = 0$

so $\mathcal{L}\{G'(t)\}(s) = F(s)$, and $\mathcal{L}\{G'(t)\}(s) = sG(s) - 0$, so $G(s) = \frac{F(s)}{s}$ ∎

$f(t)$	$F(s)$	for $t \ge 0$, $\lvert f(t)\rvert \le Ce^{Mt}$
$f(t)$	$\int_0^\infty e^{-st} f(t)\,dt$	$s > M$
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	$\mathcal{L}$ is linear!
1	$1/s$	$s > 0$
e^{at}	$1/(s-a)$	$s > \text{Re}(a)$
$\cos kt$	$s/(s^2+k^2)$	
$\sin kt$	$k/(s^2+k^2)$	
$e^{at}\cos kt$	$(s-a)/((s-a)^2+k^2)$	
$e^{at}\sin kt$	$k/((s-a)^2+k^2)$	we got this far last week.
(1a) $f'(t)$	$sF(s) - f(0)$	
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	
$f'''(t)$	$s^3F(s) - s^2f(0) - sf'(0) - f''(0)$ etc.	
$\int_0^t f(\tau)\,d\tau$	$F(s)/s$	
(1b) $tf(t)$	$-F'(s)$	
$t^2f(t)$	$F''(s)$	
$t^3f(t)$	$-F'''(s)$	
$f(t)/t$	$\int_s^\infty F(\sigma)\,d\sigma$	
(2) $e^{at}f(t)$	$F(s-a)$	
1	$1/s$	
t	$1/s^2$	
t^n	$n!/s^{n+1}$	
te^{at}	$1/(s-a)^2$	
t^ne^{at}	$n!/(s-a)^{n+1}$	
$t\cos kt$	$(s^2-k^2)/(s^2+k^2)^2$	(3) resonance!
$\frac{t}{2k}\sin kt$	$s/(s^2+k^2)^2$	
$\frac{1}{2k^3}[\sin kt - kt\cos kt]$	$1/(s^2+k^2)^2$	
$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	(4) how to $\mathcal{L}^{-1}$ products! (this is in §7.4)

(1b) (numbered with (1a) because derivatives on one side of the column corresponds to multiplication on the other side!

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$F'(s) = \lim_{\Delta s \to 0} \frac{1}{\Delta s}\left[\int_0^\infty e^{-(s+\Delta s)t} f(t)\,dt - \int_0^\infty e^{-st} f(t)\,dt\right]$$

$$= \lim_{\Delta s \to 0} \int_0^\infty f(t)\left[\frac{e^{-(s+\Delta s)t} - e^{-st}}{\Delta s}\right] dt$$

as $\Delta s \to 0$: $\frac{d}{ds} e^{-st} = -te^{-st}$

(3210 to interchange limit & integral)

$$-\int_0^\infty e^{-st}\,(tf(t))\,dt = -\mathcal{L}\{tf(t)\}(s)$$

then $\mathcal{L}\{t \cdot tf(t)\}(s) = -\frac{d}{ds}\underbrace{\mathcal{L}\{tf(t)\}(s)}_{-\frac{d}{ds}F(s)} = F''(s)$ etc. ∎

Also $\frac{d}{ds}\underbrace{\int_s^\infty F(\sigma)\,d\sigma}_{G(s)} = -F(s)$

$-G'(s) = F(s)$

$\downarrow \mathcal{L}^{-1}$

$tg(t) = f(t)$ ∎

Table entries from (1b):

$\mathcal{L}\{1\} = 1/s$ (did Wed)

$\Rightarrow \mathcal{L}\{t \cdot 1\} = -D_s\left(\frac{1}{s}\right) = \frac{1}{s^2} = s^{-2}$

$\Rightarrow \mathcal{L}\{t \cdot t\}(s) = -D_s(s^{-2}) = 2s^{-3}$

$\Rightarrow \mathcal{L}\{t \cdot t^2\}(s) = -D_s(2s^{-3}) = 3!\, s^{-4}$

$\Rightarrow$ (by induction) $\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$

also, (3) follows.

$\mathcal{L}\{\cos kt\}(s) = \frac{s}{s^2+k^2}$

$$\Rightarrow \mathcal{L}\{t\cos kt\}(s) = -D_s\left(\frac{s}{s^2+k^2}\right) = -\frac{(s^2+k^2) - s(2s)}{(s^2+k^2)^2} = \frac{s^2-k^2}{(s^2+k^2)^2}$$

$\mathcal{L}\{\sin kt\}(s) = \frac{k}{s^2+k^2}$

$$\Rightarrow \mathcal{L}\{t\sin kt\}(s) = -k(-1)(s^2+k^2)^{-2}2s = 2k\left(\frac{s}{(s^2+k^2)^2}\right)$$

$$\text{so } \mathcal{L}\left\{\frac{1}{2k} t\sin kt\right\}(s) = \frac{s}{(s^2+k^2)^2}$$

Finally, (using what we know)

$$\left.\begin{aligned} \mathcal{L}\{t\cos kt\}(s) &= \frac{s^2-k^2}{(s^2+k^2)^2} \\ \mathcal{L}\left\{\frac{t}{k}\sin kt\right\}(s) &= \frac{s^2+k^2}{(s^2+k^2)^2} \end{aligned}\right\} \mathcal{L}\left\{\frac{1}{k}\sin kt - t\cos kt\right\}(s) = 2k^2\left(\frac{1}{(s^2+k^2)^2}\right)$$

$$\text{so } \mathcal{L}\left\{\frac{1}{2k^3}\left[\sin kt - kt\cos kt\right]\right\}(s) = \frac{1}{(s^2+k^2)^2} \quad \blacksquare$$

(2) We already saw this for $\mathcal{L}\{e^{at}\cos kt\}(s) = \frac{s-a}{(s-a)^2+k^2}$

$\mathcal{L}\{e^{at}\sin kt\}(s) = \frac{k}{(s-a)^2+k^2}$

In general, $\mathcal{L}\{e^{at}f(t)\}(s) = \int_0^\infty e^{-st}e^{at}f(t)\,dt$

$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a) \quad \blacksquare$$

Table entries from (2):

$\mathcal{L}\{t^n e^{at}\}(s) = \frac{n!}{(s-a)^{n+1}}$

This just leaves (4), the convolution theorem.

First, applications of what we've done so far!

<u>example 1</u>

$$\begin{cases} x'' + 4x = 3\sin 2t \\ x(0) = 2 \\ x'(0) = 1 \end{cases}$$

Resonance! (these used to take a really long time)

$$s^2 X(s) - 2s - 1 + 4X(s) = 3\,\frac{2}{s^2+4}$$

$$X(s)\,[s^2+4] = \frac{6}{s^2+4} + 2s + 1$$

$$X(s) = \frac{6}{(s^2+4)^2} + \frac{2s}{s^2+4} + \frac{1}{s^2+4}$$

$$x(t) = 6\,\frac{1}{2\cdot 8}\left[\sin 2t - 2t\cos 2t\right] + 2\cos 2t + \tfrac{1}{2}\sin 2t$$

$$= -\frac{3}{4}t\cos 2t + \frac{7}{8}\sin 2t + 2\cos 2t$$

<u>example 2</u> §7.3 p. 466 (unforced damped spring)

$$\begin{cases} x'' + 6x' + 34x = 0 \\ x(0) = 3 \\ x'(0) = 1 \end{cases}$$

$\mathcal{L}$: $s^2 X(s) - 3s - 1 + 6\,(sX(s) - 3) + 34X(s) = 0$

$$X(s)\,(s^2 + 6s + 34) = 3s + 19$$

$$X(s) = \frac{3s+19}{s^2+6s+34}$$

$$= \frac{3(s+3) + 10}{(s+3)^2 + 25}$$

← complete the linear (?)

← complete the square

$$= 3\,\frac{(s+3)}{(s+3)^2+25} + 2\cdot\frac{5}{(s+3)^2+25}$$

↑ $3F(s+3)$ for $f(t) = \cos 5t$

↓ $G(s+3)$ for $g(t) = \sin 5t$

$\mathcal{L}^{-1}$:

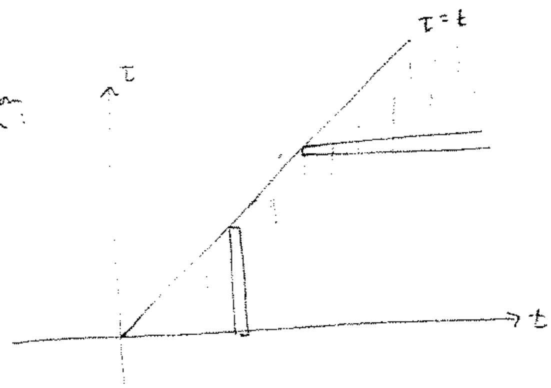
$$x(t) = 3e^{-3t}\cos 5t + 2e^{-3t}\sin 5t$$

proof of convolution theorem:
(is a good review of iterated integrals)

$$\mathcal{L}\{f*g\}(s) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(t-\tau)d\tau\right) dt$$

$$= \int_0^\infty\int_0^t e^{-st} f(\tau)g(t-\tau)\,d\tau\,dt$$

integration region:



interchange limits:

$$= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\, g(t-\tau)\, dt\, d\tau$$

$$= \int_0^\infty \int_\tau^\infty e^{-s\tau} f(\tau)\; e^{-s(t-\tau)} g(t-\tau)\,dt\,d\tau \qquad \text{(pattern recognition)}$$

$$= \int_0^\infty e^{-s\tau} f(\tau) \left[\int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)dt \right] d\tau$$

$\tilde{t} = t-\tau$

$d\tilde{t} = dt$

$$\underbrace{\left[\int_0^\infty e^{-s\tilde{t}} g(\tilde{t})\, d\tilde{t} \right]}_{G(s)}$$

$$= G(s) \int_0^\infty e^{-s\tau} f(\tau)d\tau$$

$$= G(s)\,F(s) \qquad !!$$

example: verify the theorem
for $f(t) = \sin t$
$g(t) = \cos t$
(you may need trig id $\sin^2\tau = \dfrac{1-\cos 2\tau}{2}$)

Remark:

$$f*g(t) = \int_0^t f(\tau)\, g(t-\tau)d\tau$$
$$= g*f(t)$$
(by substituting $\tilde{\tau} = t-\tau$)