Math 2270-004 Week 8 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 4.2-4.6.

Mon Feb 26

- 4.2 - 4.3 bases for vector spaces and subspaces; *Nul A* and *Col A*; generalization to linear transformations.

Announcements: I’ll post fft solts tonight

Warm-up Exercise:

for \( \mathbb{R}^n \) our favorite basis is \( \{ e_1, e_2, \ldots, e^n \} \). But for a subspace \( \mathbb{R}^n \) what's a "best" basis? no row operations required

a) check that \( \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \) is linearly independent. Which entries do you focus on?

b) express \( \begin{bmatrix} 8 \\ 2 \\ -2 \\ 3 \\ -1 \end{bmatrix} \) as a linear combination of the vectors above. Which entries do you focus on?

\[
\begin{align*}
\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} &+ c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
2^{nd} &\quad 2^{nd} entries \quad c_1 = 0 \\
4^{th} &\quad 4^{th} entries \quad c_2 = 0 \\
5^{th} &\quad 5^{th} entries \quad c_3 = 0 \\
\end{align*}
\]

Almost as good as standard basis vectors.

\[
\begin{align*}
2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ -2 \\ 3 \\ -1 \end{bmatrix} \\
\end{align*}
\]
Monday Review!

We've been discussing **vector spaces**, which are a generalization of \( \mathbb{R}^n \): Namely, a vector space is a nonempty set \( V \) of objects, called vectors, on which are defined two operations, called **addition** and **scalar multiplication**, so that ten natural axioms about vector addition and scalar multiplication hold (along with three additional useful consequences that we often use, and that you thought about on your food for thought).

Last week we discovered that certain subsets of vector spaces are also vector spaces (with the same addition and scalar multiplication as in the larger space) - namely **subspaces** of a vector space \( V \): these are subsets \( H \) of \( V \) that satisfy

a) The zero vector of \( V \) is in \( H \)

b) \( H \) is closed under vector addition, i.e. for each \( u, v \in H \) then \( u + v \in H \).

c) \( H \) is closed under scalar multiplication, i.e for each \( u \in H \), \( c \in \mathbb{R} \), then also \( c \cdot u \in H \).

We defined **linear dependence** and **linear independence** for sets of vectors \( \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \} \) in a vector space \( V \).

A basis for a vector space \( V \) is a set of vectors \( \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \} \) that span \( V \) and that is also **linearly independent**.

The dimension of a vector space \( V \) is the number of vectors in any basis for \( V \). (We'll show why every basis for a fixed vector space \( V \) - no matter how weird \( V \) may seem - has the same number of vectors, later this week.)
We showed that one way subspaces arise is as $H = \text{span}\{v_1, v_2, \ldots, v_p\}$ for sets of vectors $\{v_1, v_2, \ldots, v_p\}$ in a vector space $V$. This is an explicit way to describe $H$ because you are saying exactly which vectors are in it. If the vectors in the spanning set $\{v_1, v_2, \ldots, v_p\}$ are not already independent, we illustrated how to remove extraneous dependent vectors without shrinking the span, until we were left with a basis for the subspace $H$. (We'll return to this today .... it was an example with $H = \text{Col } A$)

We discovered that the only subsets of $\mathbb{R}^3$ that succeed at being subspaces of $\mathbb{R}^3$ are

- $\{ \mathbf{0} \}$ \hspace{1cm} 0-dimensional, by def.
- $\text{span}\{u\}$ for some $u \neq \mathbf{0}$ \hspace{1cm} 1-dimensional subspaces (a line thru the origin)
- $\text{span}\{u, v\}$ for some $\{u, v\}$ linearly independent \hspace{1cm} 2-dimensional subspaces
- $\text{span}\{u, v, w\} = \mathbb{R}^3$ for $\{u, v, w\}$ linearly independent \hspace{1cm} 3-dimensional (sub)space.

- We realized that what happens in $\mathbb{R}^3$ with respect to subspaces, generalizes to $\mathbb{R}^n$.

Towards the end of class on Friday we realized that for an $m \times n$ matrix $A$,

$$\text{Nul } A := \{ \mathbf{x} \in \mathbb{R}^n \text{ for which } A \mathbf{x} = \mathbf{0} \}$$

is a subspace. This is an implicit way to specify a subspace, because you're prescribing equations which the elements $\mathbf{x}$ must satisfy, but not explicitly saying what the elements are.

Picking up where we left off ....
Exhibit a basis for Nul(A).
4.2 Null spaces, column spaces, and linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

**Definition** Let $A$ be an $m \times n$ matrix, expressed in column form as $A = [a_1, a_2, a_3, \ldots, a_n]$. The column space of $A$, written as $Col A$, is the span of the columns:

$$Col A = span \{a_1, a_2, a_3, \ldots, a_n\}.$$ 

Equivalently, since

$$A \mathbf{x} = x_1 a_1 + x_2 a_2 + \ldots + x_n a_n,$$

we see that $Col A$ is also the range of the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\mathbf{x}) = A \mathbf{x}$, i.e

$$Col A = \{\mathbf{b} \in \mathbb{R}^m \text{ such that } \mathbf{b} = A \mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

**Theorem** By the "spans are subspaces" theorem, $Col(A)$ is always a subspace of $\mathbb{R}^m$.

**Exercise 2a)** Consider

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix}.$$ 

By the Theorem, $col(A)$ is a subspace of $\mathbb{R}^3$. Which is it: $\{0\}$, a line thru the origin, a plane thru the origin, or all of $\mathbb{R}^3$? Hint:

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix} \text{ reduces to } \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

2b) Is there a more efficient way to express $Col A$ as a span that doesn't require all five column vectors?

$$c_1 \tilde{a}_1 + c_2 \tilde{a}_2 + c_3 \tilde{a}_3 + c_4 \tilde{a}_4 + c_5 \tilde{a}_5$$

$$= c_1 \tilde{a}_1 + c_2 (-2 \tilde{a}_2) + c_3 \tilde{a}_3 + c_4 (-\tilde{a}_1 + \tilde{a}_3) + c_5 (\tilde{a}_1 + \tilde{a}_3)$$

$$= d_1 \tilde{a}_1 + d_3 \tilde{a}_3 \quad \text{Col } A = \text{span } \{\tilde{a}_1, \tilde{a}_3, \tilde{a}_5\} \quad \text{span } \{\tilde{a}_1, \tilde{a}_3\}$$

so, $col A$ was just a plane thru origin.
Not all bases are created equal!

**Theorem:** Let \( \text{span} \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \} = H \) be a subspace. The following *elementary operations* do not effect the span of the resulting ordered set:

(i) swap two of the vectors in the set, i.e. replace \( \mathbf{v}_j \) with \( \mathbf{v}_k \), and replace \( \mathbf{v}_k \) with \( \mathbf{v}_j \).

\[
\text{span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_j, \ldots, \mathbf{v}_k, \ldots, \mathbf{v}_p \} = \text{span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_k, \ldots, \mathbf{v}_j, \ldots, \mathbf{v}_p \}
\]

(ii) replace \( \mathbf{v}_j \) with \( c \mathbf{v}_j \), for \( c \neq 0 \).

\[
\text{span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_j, \ldots, \mathbf{v}_p \} = \text{span} \{ \mathbf{v}_1, \ldots, c \mathbf{v}_j, \ldots, \mathbf{v}_p \}
\]

(iii) for \( j \neq k \), replace \( \mathbf{v}_k \) with \( \mathbf{v}_k + c \mathbf{v}_j \).

\[
\text{span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_j, \ldots, \mathbf{v}_k, \ldots, \mathbf{v}_p \} = \text{span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_j, \ldots, \mathbf{v}_k + c \mathbf{v}_j, \ldots, \mathbf{v}_p \}
\]

**Exercise 1** Use the "change of spanning set" theorem above, to find a better basis for \( \text{Col} \ A \) then the one we came up with by culling dependent vectors, on Friday. Hint: Use elementary column operations to compute the reduced column echelon form of \( A \). Illustrate why this new basis is a better basis for \( \text{Col} \ A \) by seeing how easy it is to express any one of the original column vectors in terms of this improved basis.

In this example, \( A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix} \) and \( \text{Col} \ A = \text{span} \{ a_1, a_2, a_3, a_4, a_5 \} = \text{span} \{ \overline{a_1}, \overline{a_2} \} \).

\[
A = \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix}.
\]

As we just reviewed, on Friday we realized that a pretty good basis for \( \text{Col} \ A \) is \( \{ a_1, a_3 \} \):

\[
\begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix} \quad \text{row reduces to} \quad \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]
Now column reduce $A$ to get a basis for $\text{Col} A$ that's as good as you could hope for....and show this by expressing each of the original columns in terms of this basis.

$$A = \begin{bmatrix}
1 & -2 & 0 & -1 & 1 \\
-2 & 4 & 0 & 2 & -2 \\
3 & -6 & 4 & 1 & 7
\end{bmatrix}$$

Let's reduce $A$ to get a basis for $\text{Col} A$. Here is the process:

1. $2\vec{a}_1 + \vec{a}_2 \rightarrow \vec{a}_2$
2. $\vec{a}_1 + \vec{a}_4 \rightarrow \vec{a}_4$
3. $-\vec{a}_1 + \vec{a}_3 \rightarrow \vec{a}_3$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 \\
3 & 0 & 4 & 4 & 4
\end{bmatrix}$$

4. $\vec{a}_2 \rightarrow \vec{a}_5$
5. $\vec{a}_3 \rightarrow \vec{a}_2$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 \\
3 & 0 & 4 & 4 & 0
\end{bmatrix}$$

6. $\frac{1}{2} \vec{a}_2 \rightarrow \vec{a}_6$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 \\
3 & 1 & 4 & 4 & 0 & 0
\end{bmatrix}$$

7. $-3\vec{a}_2 + \vec{a}_3 \rightarrow \vec{a}_7$
8. $-4\vec{a}_2 + \vec{a}_4 \rightarrow \vec{a}_8$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Tuesday:

$$\vec{a}_3 = \begin{bmatrix} 1 \\ -2 \\ 7 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{a}_4 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
general linear transformations.

The ideas of nullspace and column space generalize to arbitrary linear transformations between vectorspaces - with slightly more general terminology.

**Definition** Let \( V \) and \( W \) be vector spaces. A function \( T : V \to W \) is called a **linear transformation** if for each \( \mathbf{x} \in V \) there is a unique vector \( T(\mathbf{x}) \in W \) and so that

- (i) \( T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \) for all \( \mathbf{u}, \mathbf{v} \in V \)
- (ii) \( T(c \mathbf{u}) = c T(\mathbf{u}) \) for all \( \mathbf{u} \in V, c \in \mathbb{R} \)

**Definition** The **kernel** (or **nullspace**) of \( T \) is defined to be \( \{ \mathbf{u} \in V : T(\mathbf{u}) = \mathbf{0} \} \).

**Definition** The **range** of \( T \) is \( \{ \mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V \} \).

**Theorem** Let \( T : V \to W \) be a linear transformation. Then the kernel of \( T \) is a subspace of \( V \). The range of \( T \) is a subspace of \( W \).

\[
\text{Range } T \text{ is subspace of } W : \quad 1) \text{ is } \mathbf{0} \in \text{Range } T \implies \mathbf{0} \in W \text{ (show 1), 2, 3).} \\
2) \text{ is } \text{Range } T \text{ closed under addition?} \\
3) \text{ is } \text{Range } T \text{ closed under scalar multiplication?}
\]

\[
\text{Let } \mathbf{w}_1, \mathbf{w}_2 \in \text{Range } T \Rightarrow T(\mathbf{v}_1) = \mathbf{w}_1, T(\mathbf{v}_2) = \mathbf{w}_2 \text{ then } T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2) \text{ (true)} \\
\text{let } \lambda \mathbf{v} \in \text{Range } T \text{ is } \lambda \mathbf{v} \in \text{Range } T \text{ (true)}
\]
Exercise 2  Let $V$ be the vector space $C^1[a, b]$ of real-valued functions $f$ defined on an interval $[a, b]$ with the property that they are differentiable and that their derivatives are continuous functions on $[a, b]$. Let $W$ be the vector space $C[a, b]$ of all continuous functions on the interval $[a, b]$. Let $D : V \rightarrow W$ be the derivative transformation

$$D(f) = f'$$

2a) What Calculus differentiation rules tell you that $D$ is a linear transformation?

$$(f + g)' = f' + g'$$
$$D(f + g) = D(f) + D(g)$$
$$(cf)' = cf'$$
$$D(cf) = cD(f)$$

2b) What subspace is the kernel of $D$?

$$\text{kernel } D = \{ \text{constant functions} \} = \{ c, c \in \mathbb{R} \}$$
$$\text{Mean Value Theorem}$$

$$= \text{span } \{ 1 \}$$
$$1(x) = 1$$

2c) What is the range of $D$?

$$= C[a, b]$$

because if $f(x)$ is continuous on $[a, b]$
then $\int_a^x f(t) dt$ is an antiderivative $\in C^1[a, b]$.

$$F(x) = \int_a^x f(t) dt$$
$$D(F) = f$$

Fundamental Theorem of Calculus, part I
4.4 Coordinate systems for finite dimensional vector spaces

Announcements: Let $V$ be a vector space with basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.
This basis gives us a coordinate system for $V$:
$$\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n$$
$c_1, c_2, \ldots, c_n$ are the coordinates of $\mathbf{v}$ with respect to $B$.

Warm-up Exercise:
consider $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}\right\}$ which is a plane in $\mathbb{R}^3$ with implicit equation $0x + 2y + z = 0$.

the point $\begin{bmatrix} 1.5 \\ .5 \\ -1 \end{bmatrix}$ lies on this plane $(2y + z = 0)$

Find $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ so that
$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ .5 \\ -1 \end{bmatrix}.$$ 

Sketch $1.5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + .5 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$
“coordinates of $\begin{bmatrix} 1.5 \\ .5 \\ -1 \end{bmatrix}$ with respect to $B$ are $[1.5]$”

$$\begin{bmatrix} 1.5 \\ .5 \end{bmatrix}$$

this is the way that a plane than the origin in $\mathbb{R}^3$ is “like” $\mathbb{R}^2$, but not $\mathbb{R}^2$. 


Theorem  Let \( V \) be a vector space, and let \( \{ b_1, b_2, \ldots, b_p \} \) be a basis for \( V \). Then for each \( v \in V \) there is a unique set of scalars \( c_1, c_2, \ldots, c_p \) so that
\[
v = c_1 b_1 + c_2 b_2 + \ldots + c_p b_p.
\]

proof: Suppose we could also write
\[
\vec{v} = d_1 b_1 + d_2 b_2 + \ldots + d_p b_p.
\]

Subtract \( E_2 \) from \( E_1 \)
\[
\overrightarrow{0} = c_1 b_1 + c_2 b_2 + \ldots + c_p b_p - (d_1 b_1 + d_2 b_2 + \ldots + d_p b_p)
\]
\[
= (c_1 - d_1) b_1 + (c_2 - d_2) b_2 + \ldots + (c_p - d_p) b_p.
\]

because \( \{ b_1, b_2, \ldots, b_p \} \) in lin. indep.

know \( c_1 - d_1 = 0 \)
\[
c_2 - d_2 = 0
\]
\[
\vdots
\]
\[
c_p - d_p = 0
\]

i.e. each \( c_j = d_j \).
Definition (Each basis gives us a coordinate system). Let $V$ be a vector space, and let $\beta = \{ b_1, b_2, \ldots b_p \}$ be a basis for $V$. For each $v \in V$ we say that the coordinates of $v$ with respect to $\beta$ are $c_1, c_2, \ldots c_p$ if

$$v = c_1 b_1 + c_2 b_2 + \ldots + c_p b_p.$$ 

And, we write the vector of the coordinates of $v$ with respect to $\beta$ as:

$$[v]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix} \in \mathbb{R}^p.$$ 

Example: For the vector space

$$P_3 = \{ p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \text{ such that } a_0, a_1, a_2, a_3 \in \mathbb{R} \}$$

we've checked that

$$\beta = \{ 1, t, t^2, t^3 \}$$

is a basis. So the coordinate vector of

$$p(t) = 3 - 4 t^2 + t^3 \quad \cdot$$

with respect to $\beta$ is

$$[p]_\beta = \begin{bmatrix} 3 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$
And, if \( q \in P_3 \), with

\[
[q]_\beta = \begin{bmatrix} -2 \\ 1 \\ 7 \\ 0 \end{bmatrix}
\]

then

\[ q(t) = -2 + t + 7 t^2. \]

It turns out that we can understand pretty much any vector space question about \( P_3 \) by interpreting the question in terms of the coordinates with respect to \( \beta \), which lets us work in \( \mathbb{R}^4 \) in lieu of \( P_3 \). That's what coordinates with respect to a basis are good for, when you're working with a non-standard vector space.
Exercise 1) Let \( \beta = \{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \} = \{ \mathbf{u}, \mathbf{v} \} \) be a non-standard basis of \( \mathbb{R}^2 \).

1a) Suppose \( \mathbf{x} \) is a vector in \( \mathbb{R}^2 \), and 
\[
[\mathbf{x}]_\beta = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.
\]
Find the standard coordinates for \( \mathbf{x} \), i.e. its coordinates with respect to the standard basis \( E = \{ \mathbf{e}_1, \mathbf{e}_2 \} \):
\[
\mathbf{x} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = [\mathbf{x}]_E \quad \overset{\text{transition matrix}}{\implies} \quad [\mathbf{x}]_E = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} [\mathbf{x}]_\beta
\]

1b) Find the \( \beta \)-coordinates for the vector \( \mathbf{b} = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \). (The math may seem familiar.)

1c) Interpret your work in 1ab geometrically, in terms of the coordinate system generated by \( \beta \).
Theorem Let \( V, W \) be vector spaces, and \( T : V \to W \) a linear transformation. If \( T \) is \( 1 \)–\( 1 \) and onto, then the inverse function \( T^{-1} \) is also a linear transformation, \( T^{-1} : W \to V \). In this case, we call \( T \) an isomorphism.

**proof:** We have to check that for all \( \mathbf{u}, \mathbf{w} \in W \) and all \( c \in \mathbb{R} \),

\[
T^{-1} (\mathbf{u} + \mathbf{w}) = T^{-1} (\mathbf{u}) + T^{-1} (\mathbf{w})
\]

\[
T^{-1} (c \mathbf{u}) = c \cdot T^{-1} (\mathbf{u}).
\]

Since \( T \) is \( 1\)-\( 1 \), it suffices to check that \( T \) satisfies LHS’s = RHS’s.

\[
T_{\mathbf{u}} \text{ RHS's \: } T (T^{-1}(\mathbf{u}) + T^{-1}(\mathbf{w})) = T (T^{-1}(\mathbf{u})) + T (T^{-1}(\mathbf{w}))
\]

\[
T_{\mathbf{u}} \text{ LHS's \: } T (T^{-1}(\mathbf{u} + \mathbf{w})) = \mathbf{u} + \mathbf{w}
\]

For scalar multiplication:

\[
T (T^{-1}(c \mathbf{u})) = c \cdot T (T^{-1}(\mathbf{u})) = c \cdot \mathbf{u}
\]

Theorem Let \( V \) be a vector space, with basis \( \beta = \{ \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \} \). Then the coordinate transform \( T : V \to \mathbb{R}^n \) defined by

\[
T (\mathbf{v}) = [\mathbf{v}]_{\beta}
\]

is linear, and it is an isomorphism.
Exercise 2: Use coordinates with respect to the basis \( \{1, t, t^2\} \), to check whether or not the set of polynomials \( \{p_1(t), p_2(t), p_3(t)\} \) is a basis for \( P_2 \), where

\[
P_1(t) = 1 + t^2 \quad \text{etc.}
P_2(t) = 2 + 3t + t^2
P_3(t) = -3t + t^2.
\]

i.e. check whether the coordinate vectors in \( \mathbb{R}^3 \) are independent & span \( \mathbb{R}^3 \). Then make conclusions about \( \{p_1, p_2, p_3\} \):

\[
p_2 = 2p_1 - p_3
2p_1 - p_2 - p_3 = 0
2(t + 1) - (2 + 3t + t^2) - (3t + t^2) = 0.
\]

hour, with coordinate vectors instead the "old way"

\[
\begin{bmatrix}
1 & 0 & 1
\end{bmatrix}
+ \begin{bmatrix}
2 & 3 & -3
\end{bmatrix}
+ \begin{bmatrix}
0 & -1 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0
\end{bmatrix}
\]

\[
P_1(t) = 1 + 0t + 1t^2
P_2(t) = 2 + 3t + t^2
\]

\[
\begin{bmatrix}
1 & 2 & 0
0 & 3 & -3
1 & 1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 2
0 & 1 & -1
0 & 0 & 0
\end{bmatrix}
\]

ref

\[
\begin{bmatrix}
c_1 & c_2 & c_3
\end{bmatrix}
= \begin{bmatrix}
-2 & 3 & 1
\end{bmatrix}
\]

\[
c_1 = -2c_3
c_2 = c_3
\]

\[
c_3 = \text{free}
\]

**Theorem**: If \( T: V \rightarrow W \) is an isomorphism, then \( \{v_1, \ldots, v_p\} \) is (in)dependent in \( V \) if and only if \( \{Tv_1, \ldots, Tv_p\} \) are (in)dependent in \( W \)

**proof**: If \( c_1v_1 + c_2v_2 + \cdots + c_pv_p = \overrightarrow{0} \)
then \( T(c_1v_1 + c_2v_2 + \cdots + c_pv_p) = T(\overrightarrow{0}) = \overrightarrow{0} \)

by linearity

\[
c_1T(v_1) + c_2T(v_2) + \cdots + c_pT(v_p) = \overrightarrow{0}
\]

So, if \( \{v_1, \ldots, v_p\} \) are dependent then \( \{Tv_1, Tv_2, \ldots, Tv_p\} \) are also dependent with the same weights.

by logic: same fact for "independent"
Exercise 3  Generalize the example of Exercise 1: Suppose $\mathbf{B} = \{ \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \}$ is a non-standard basis of $\mathbb{R}^n$. And let $E = \{ \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \}$ be the standard basis of $\mathbb{R}^n$. For $\mathbf{x} \in \mathbb{R}^n$, how do you convert between $\mathbf{x} = [\mathbf{x}]_E$ and $[\mathbf{x}]_B$, and vise-verse?

\[
\begin{aligned}
\forall \mathbf{x} &\in \mathbb{R}^n, \quad \mathbf{x} = [\mathbf{x}]_E = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 + \ldots + c_n \mathbf{b}_n = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \\
\text{that means} \quad [\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \\
\forall \mathbf{x} &\in \mathbb{R}^n, \quad [\mathbf{x}]_E = \mathbf{B} [\mathbf{x}]_B \quad \text{B is called} \quad P_{E \leftarrow B} \\
\forall \mathbf{x} &\in \mathbb{R}^n, \quad \mathbf{B}^{-1} [\mathbf{x}]_E = [\mathbf{x}]_B \quad \text{B}^{-1} \text{is called} \quad P_{B \leftarrow E}
\end{aligned}
\]
Wed Feb 28
- 4.5 dimension of a vector space, and related facts about span and linear independence.

Announcements:
- Tuesday's notes today.
- Quiz

Warm-up Exercise:

\[
\begin{bmatrix}
1 & 3 & 0 & 4 \\
2 & -2 & -8 & 0 \\
-1 & 2 & 5 & 1 \\
0 & 4 & 4 & 4
\end{bmatrix}
\]

\[
\text{rref}(A) = \begin{bmatrix}
1 & 0 & -3 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

what is the dimension of \( \text{Col } A \)? = 2  (# of vectors in a basis)

a basis for \( \text{Col } A \)? \( \{ \vec{a}_1, \vec{a}_2 \} \)

what is dimension of \( \text{Nul } A \)?

could you find a basis of \( \text{Nul } A \)?

\[
\text{dim } \text{Nul } A = 2 = \# \text{ of free variables when we solve } \Delta \vec{x} = \vec{b}
\]

= \# columns of \text{rref}(A) without pivots,
There is a circle of ideas related to linear independence, span, and bases for vector spaces, which it is good to try and understand carefully. That's what we'll do today. These ideas generalize (and use) ideas we've already explored more concretely, and facts we already know to be true for the vector spaces $\mathbb{R}^n$.

**Theorem 1** (constructing a basis from a spanning set): Let $V$ be a vector space of dimension at least one, and let $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} = V$. Then a subset of the spanning set is a basis for $V$. (We followed a procedure like this to extract bases for $\text{Col} A$.)

**Theorem 2** Let $V$ be a vector space, with basis $\beta = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$. Then any set in $V$ containing more than $n$ elements must be linearly dependent. (We used reduced row echelon form to understand this in $\mathbb{R}^n$.)
Theorem 3. Let $V$ be a vector space, with basis $\beta = \{b_1, b_2, \ldots, b_n\}$. Let $\alpha = \{a_1, a_2, \ldots, a_p\}$ be a set of independent vectors that don't span $V$. Then $p < n$, and additional vectors can be added to the set $\alpha$ to create a basis $\{a_1, a_2, \ldots, a_p, \ldots, a_n\}$ (We followed a procedure like this when we figured out all the subspaces of $\mathbb{R}^3$.)

Theorem 4. Let $V$ be a vector space, with basis $\beta = \{b_1, b_2, \ldots, b_n\}$. Then no set $\alpha = \{a_1, a_2, \ldots, a_p\}$ with $p < n$ vectors can span $V$. (We know this for $\mathbb{R}^n$.)
**Theorem 5** Let $V$ be a vector space, with basis $\beta = \{b_1, b_2, \ldots, b_n\}$. Then every basis for $V$ has exactly $n$ vectors. Furthermore, if $\alpha = \{a_1, a_2, \ldots, a_n\}$ is another collection of exactly $n$ vectors in $V$, and if $\text{span}\{a_1, a_2, \ldots, a_n\} = V$, then the set $\alpha$ is automatically linearly independent and a basis. Conversely, if the set $\{a_1, a_2, \ldots, a_n\}$ is linearly independent, then $\text{span}\{a_1, a_2, \ldots, a_n\} = V$ is guaranteed, and $\alpha$ is a basis. (We know all these facts for $\mathbb{R}^n$ from reduced row echelon form considerations.)

**Corollary** Let $V$ be a vector space of dimension $n$. Then the subspaces of $V$ have dimensions $0, 1, 2, \ldots, n - 1, n$. (We know this for $\mathbb{R}^n$.)
Fri Mar 2
• 4.6 The four subspaces associated with a matrix. the rank of a matrix.

Announcements:

Finish Tuesday notes
I think we'll finish Wed, but if we don't, that's fine

Warm-up Exercise:

See Tuesday notes
Theorem Let $V, W$ be vector spaces, and $T : V \to W$ a linear transformation. If $T$ is 1–1 and onto, then the inverse function $T^{-1}$ is also a linear transformation, $T^{-1} : W \to V$. In this case, we call $T$ an isomorphism.

proof: We have to check that for all $u, w \in W$ and all $c \in \mathbb{R}$,

\[ T^{-1}(u + w) = T^{-1}(u) + T^{-1}(w) \]
\[ T^{-1}(cu) = cT^{-1}(u). \]

Since $T$ is 1-1, it suffices to check that $T \circ \text{LHS}'s = T \circ \text{RHS}'s$

\[ T \circ \text{LHS} : T^{-1}(\bar{u}) + T^{-1}(\bar{w}) \]
\[ = \bar{u} + \bar{w} \]
\[ = T(\bar{u} + \bar{w}) \]
\[ = T \circ \text{LHS} : \]
\[ T \circ \text{RHS} : \]
\[ = T^{-1}(\bar{u} + \bar{w}) \]
\[ = T^{-1}(\bar{u}) + T^{-1}(\bar{w}) \]

for scalar multiplication:
\[ T(T^{-1}(\bar{u})) = c \bar{u}, \quad T(cT^{-1}(\bar{u})) = cT^{-1}(\bar{u}) \]

Theorem Let $V$ be a vector space, with basis $\beta = \{b_1, b_2, \ldots, b_n\}$. Then the coordinate transform $T : V \to \mathbb{R}^n$ defined by

\[ T(v) = [v]_\beta \]

is linear, and it is an isomorphism.

---

\[ (1) \quad T(v + w) = \begin{bmatrix} c_1 + d_1 \\ c_2 + d_2 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} \]
\[ [v + w]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} \]

(1) \quad $T(v + w) = T(v) + T(w)$

\[ T(cv) = \begin{bmatrix} c_1 + d_1 \\ c_2 + d_2 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} \]

(2) \quad $T(cv) = cT(v)$

I know $T^{-1} : \mathbb{R}^n \to V$

\[ T^{-1}\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = e_1 \bar{v}_1 + e_2 \bar{v}_2 + \ldots + e_n \bar{v}_n. \]
Exercise 2: Use coordinates with respect to the basis \( \{1, t, t^2\} \), to check whether or not the set of polynomials \( \{p_1(t), p_2(t), p_3(t)\} \) is a basis for \( P_2 \), where

\[
\begin{align*}
p_1(t) &= 1 + t^2 \\
p_2(t) &= 2 + 3t + t^2 \\
p_3(t) &= -3t + t^2.
\end{align*}
\]

i.e. check whether the coord vectors in \( \mathbb{R}^3 \) are independent & span \( \mathbb{R}^3 \). Then make conclusions about \( \{p_1, p_2, p_3\} \)

\[
\begin{align*}
p_2 &= 2p_1 - p_3 \\
2p_1 - p_2 - p_3 &= 0 \\
2(\lambda + t) - (2 + 3t + t^2) - (-3t + t^2) &= 0.
\end{align*}
\]

show, with coord vectors instead the "old way"

\[
\begin{align*}
&c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
c_1 [1, 0, 1] + c_2 [2, 3, 1] + c_3 [0, 1, 1] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{align*}
\]

\[
\begin{array}{ccc|c}
1 & 2 & 0 & 0 \\
0 & 3 & -3 & 0 \\
1 & 1 & 1 & 0
\end{array} \xrightarrow{\text{ref}}
\begin{array}{ccc|c}
1 & 2 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{array}
\]

\[
c_1 = -2c_3 \\
c_2 = c_3 \\
c_3 = \text{free}
\]

**Theorem:** If \( T: V \rightarrow W \) is an isomorphism, then \( \{\vec{v}_1, \ldots, \vec{v}_p\} \) is (in)dependent in \( V \) if and only if \( \{T\vec{v}_1, T\vec{v}_2, \ldots, T\vec{v}_p\} \) are (in)dependent in \( W \)

**Proof:**

If \( c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p = \overrightarrow{0} \)

then \( T(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p) = T(\overrightarrow{0}) = \overrightarrow{0} \)

by linearity

so, if \( \{\vec{v}_1, \ldots, \vec{v}_p\} \) are dependent then \( \{T\vec{v}_1, T\vec{v}_2, \ldots, T\vec{v}_p\} \) are also dependent, with the same weights

by logic:

same fact

for "independent"
Exercise 3  Generalize the example of Exercise 1: Suppose $\beta = \{b_1, b_2, \ldots, b_n\}$ is a non-standard basis of $\mathbb{R}^n$. And let $E = \{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, how do you convert between $x = [x]_E$ and $[x]_\beta$ and vise-verse?

- If $x = [x]_E = c_1 b_1 + c_2 b_2 + \ldots + c_n b_n = \begin{bmatrix} b_1 & b_2 & \ldots & b_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

  that means $[x]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

- $[x]_E = B [x]_\beta$  
  $B$ is called $P_{E \leftrightarrow \beta}$

- $B^{-1} [x]_E = [x]_\beta$  
  $B^{-1}$ is called $P_{\beta \leftrightarrow E}$