Math 2270-004 Week 8 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 4.2-4.6.

Mon Feb 26

- 4.2 - 4.3 bases for vector spaces and subspaces; *Nul A* and *Col A*; generalization to linear transformations.

Announcements:

Warm-up Exercise:

Monday Review!

We've been discussing *vector spaces*, which are a generalization of $\mathbb{R}^n$: Namely, a *vector space* is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *scalar multiplication,* so that ten natural axioms about vector addition and scalar multiplication hold (along with three additional useful consequences that we often use, and that you thought about on your food for thought).

Last week we discovered that certain subsets of vector spaces are also vector spaces (with the same addition and scalar multiplication as in the larger space) - namely *subspaces* of a vector space V: these are subsets H of V that satisfy

a) The zero vector of V is in H
b) H is closed under vector addition, i.e. for each $\underline{\boldsymbol{u}} \in H, \underline{\boldsymbol{v}} \in H$ then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in H$.
c) H is closed under scalar multiplication, i.e for each $\underline{\boldsymbol{u}} \in H$, $c \in \mathbb{R}$, then also $c\,\underline{\boldsymbol{u}} \in H$.

We defined *linear dependence* and *linear independence* for sets of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$ in a vector space V.

A *basis* for a vector space V is a set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$ that *span* V and that is also *linearly* independent.

The *dimension* of a vector space V is the number of vectors in any basis for V. (We'll show why every basis for a fixed vector space V - no matter how weird V may seem - has the same number of vectors, later this week.)

We showed that one way subspaces arise is as $H = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ \ldots \ \underline{\boldsymbol{v}}_p\}$ for sets of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ \ldots \ \underline{\boldsymbol{v}}_p\}$ in a vector space V. This is an explicit way to describe H because you are saying exactly which vectors are in it. If the vectors in the spanning set $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ \ldots \ \underline{\boldsymbol{v}}_p\}$ are not already independent, we illustrated how to remove extraneous dependent vectors without shrinking the span, until we were left with a basis for the subspace H. (We'll return to this today .... it was an example with $H = Col\ A$)

We discovered that the only subsets of $\mathbb{R}^3$ that succeed at being subspaces of $\mathbb{R}^3$ are

- $\{\ \underline{\mathbf{0}}\ \}$
- $span\{\underline{\boldsymbol{u}}\}$ for some $\underline{\boldsymbol{u}} \neq \underline{\mathbf{0}}$ (a line thru the origin) 1 - dimensional subspaces
- $span\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}\}$ for some $\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}\}$ linearly independent 2 - dimensional subspaces
- $span\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}}\} = \mathbb{R}^3$ for $\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}}\}$ linearly independent 3 - dimensional (sub)space.

We realized that what happens in $\mathbb{R}^3$ with respect to subspaces, generalizes to $\mathbb{R}^n$.

Towards the end of class on Friday we realized that for an $m \times n$ matrix A,

$$Nul\ A := \{\underline{\boldsymbol{x}} \in \mathbb{R}^n \text{ for which } A\,\underline{\boldsymbol{x}} = \underline{\mathbf{0}}\}$$

is a subspace. This is an implicit way to specify a subspace, because you're prescribing equations which the elements $\underline{\boldsymbol{x}}$ musts satisfy, but not explicitly saying what the elements are.

Picking up where we left off ....

Monday Feb. 26
finish from Friday

Exercise 1a) For the same matrix A as in Exercise 2 from Wednesday's notes, express the vectors in $Nul(A)$ explicitly, using the methods of Chapters 1-2. Notice these are vectors in the domain of the associated linear transformation $T : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ given by $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$, so are a subspace of $\mathbb{R}^5$.

$A\vec{x} = \vec{0}$

$$A = \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 1 & 0 \\ -2 & 4 & 0 & 2 & -2 & 0 \\ 3 & -6 & 4 & 1 & 7 & 0 \end{array}\right] \text{ reduces to } \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \leftarrow$$

$\{\vec{x} : A\vec{x} = \vec{0}\}$

implicit description

$x_1 = 2t_2 + t_4 - t_5$

$x_2 = t_2$ free

$x_3 = -t_4 - t_5$

$x_4 = t_4$ free

$x_5 = t_5$ free

explicit description

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = t_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t_5 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

1b) Exhibit a basis for $Nul(A)$.

$$Nul\ A = span\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

actually a basis

Monday Feb 26
Review

from Wed. notes, covered Friday.

4.2 Null spaces, column spaces, and linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Definition Let A be an $m \times n$ matrix, expressed in column form as $A = [\underline{a}_1\ \underline{a}_2\ \underline{a}_3\ \dots\ \underline{a}_n]$ The *column space* of A, written as $Col\,A$, is the span of the columns:

$$Col\,A = span\{\underline{a}_1\ \underline{a}_2\ \underline{a}_3\ \dots\ \underline{a}_n\}.$$

Equivalently, since

$$A\,\underline{x} = x_1\,\underline{a}_1 + x_2\,\underline{a}_2 + \dots + x_n\,\underline{a}_n$$

we see that $Col\,A$ is also the range of the linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ given by $T(\underline{x}) = A\,\underline{x}$, i.e

$$Col\,A = \{\underline{b} \in \mathbb{R}^m \text{ such that } \underline{b} = A\,\underline{x} \text{ for some } \underline{x} \in \mathbb{R}^n\}.$$

Theorem By the "spans are subspaces" theorem, $Col(A)$ is always a subspace of $\mathbb{R}^m$.

Exercise 2a) Consider

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix}.$$

By the Theorem, $col(A)$ is a subspace of $\mathbb{R}^3$. Which is it: $\{\underline{0}\}$, a line thru the origin, a plane thru the origin, or all of $\mathbb{R}^3$. Hint:

$$\begin{array}{ccccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \\ \end{array} \qquad\qquad \begin{array}{ccccc} \vec{r}_1 & \vec{r}_2 & \vec{r}_3 & \vec{r}_4 & \vec{r}_5 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix} \quad \text{reduces to} \quad \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\vec{a}_1 \neq \vec{0}$		$\vec{r}_1 \neq \vec{0}$
$\vec{a}_2 = -2\vec{a}_1$		$\vec{r}_2 = -2\vec{r}_1$
$\vec{a}_3$ ind of $\vec{a}_1$	$\Longleftarrow$	$\vec{r}_3$ ind. of $\vec{r}_1$
$\vec{a}_4 = -\vec{a}_1 + \vec{a}_3$		$\vec{r}_4 = -\vec{r}_1 + \vec{r}_3$
$\vec{a}_5 = \vec{a}_1 + \vec{a}_3$		$\vec{r}_5 = \vec{r}_1 + \vec{r}_3$

2b) Is there a more efficient way to express $Col\,A$ as a span that doesn't require all five column vectors?

YES.

$$c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 + c_4\vec{a}_4 + c_5\vec{a}_5$$
$$= c_1\vec{a}_1 + c_2(-2\vec{a}_1) + c_3\vec{a}_3 + c_4(-\vec{a}_1 + \vec{a}_3) + c_5(\vec{a}_1 + \vec{a}_3)$$
$$= d_1\vec{a}_1 + d_3\vec{a}_3$$

$$Col\,A = span\{\vec{a}_1, \vec{a}_2, \dots \vec{a}_5\} = span\{\vec{a}_1, \vec{a}_3\}$$

so, col A was just a plane thru origin

Not all bases are created equal!

Theorem: Let $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\} = H$ be a subspace. The following *elementary operations* do not effect the span of the resulting ordered set:

(i) swap two of the vectors in the set, i.e. replace $\underline{\boldsymbol{v}}_j$ with $\underline{\boldsymbol{v}}_k$, and replace $\underline{\boldsymbol{v}}_k$ with $\underline{\boldsymbol{v}}_j$.

(ii) replace $\underline{\boldsymbol{v}}_j$ with $c\,\underline{\boldsymbol{v}}_j$, for $c \neq 0$.

(iii) for $j \neq k$, replace $\underline{\boldsymbol{v}}_k$ with $\underline{\boldsymbol{v}}_k + c\,\underline{\boldsymbol{v}}_j$.

Exercise 1) Use the "change of spanning set" theorem above, to find a better basis for $Col\,A$ then the one we came up with by culling dependent vectors, on Friday. Hint: Use elementary column operations to compute the reduced column echelon form of A. Illustrate why this new basis is a better basis for $Col\,A$ by seeing how easy it is to express any one of the original column vectors in terms of this improved basis.

In this example, $A = [\underline{\boldsymbol{a}}_1\ \underline{\boldsymbol{a}}_2\ \underline{\boldsymbol{a}}_3\ \underline{\boldsymbol{a}}_4\ \underline{\boldsymbol{a}}_5]$ and $Col\,A = span\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \underline{\boldsymbol{a}}_3, \underline{\boldsymbol{a}}_4, \underline{\boldsymbol{a}}_5\}$

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix}.$$

As we just reviewed, on Friday we realized that a pretty good basis for $Col\,A$ is $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_3\}$:

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix} \quad \text{row reduces to} \quad \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now column reduce A to get a basis for $Col\ A$ that's as good as you could hope for....and show this by expressing each of the original columns in terms of this basis.

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix}$$

general linear transformations.

The ideas of nullspace and column space generalize to arbitrary linear transformations between vectors spaces - with slightly more general terminology.

Definition Let V and W be vector spaces. A function $T: V \to W$ is called a *linear transformation* if for each $\underline{\mathbf{x}} \in V$ there is a unique vector $T(\underline{\mathbf{x}}) \in W$ and so that

(i) $T(\underline{\mathbf{u}} + \underline{\mathbf{v}}) = T(\underline{\mathbf{u}}) + T(\underline{\mathbf{v}})$ for all $\underline{\mathbf{u}}, \underline{\mathbf{v}} \in V$

(ii) $T(c\,\underline{\mathbf{u}}) = c\,T(\underline{\mathbf{u}})$ for all $\underline{\mathbf{u}} \in V$, $c \in \mathbb{R}$

Definition The *kernel* (or *nullspace)* of T is defined to be $\{\underline{\mathbf{u}} \in V:\ T(\underline{\mathbf{u}}) = \underline{\mathbf{0}}\}$.

Definition The *range* of T is $\{\underline{\mathbf{w}} \in W:\ \underline{\mathbf{w}} = T(\underline{\mathbf{v}})\ \textit{for some}\ \underline{\mathbf{v}} \in V\}$.

Theorem Let $T: V \to W$ be a *linear transformation.* Then the kernel of T is a subspace of V. The range of T is a subspace of W.

Remark: The theorem generalizes our earlier one about *Nul A* and *Col A*, for matrix transformations $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$.

Exercise 2 Let V be the vector space $C^1[a, b]$ of real-valued functions f defined on an interval $[a, b]$ with the property that they are differentiable and that their derivatives are continuous functions on $[a, b]$. Let W be the vector space $C[a, b]$ of all continous functions on the interval $[a, b]$. Let $\mathrm{D} : V \rightarrow W$ be the derivative transformation

$$\mathrm{D}(f) = f' .$$

2a) What Calculus differentiation rules tell you that D is a linear transformation?

2b) What subspace is the kernel of D ?

2c) What is the range of D ?

Tues Feb 27

- 4.4 Coordinate systems for finite dimensional vector spaces

Announcements:

Warm-up Exercise:

Theorem Let V be a vector space, and let $\{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ \ldots \underline{\boldsymbol{b}}_p\}$ be a basis for V. Then for each $\underline{\boldsymbol{v}} \in V$ there is a unique set of scalars $c_1, c_2, \ldots c_p$ so that

$$\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{b}}_1 + c_2\underline{\boldsymbol{b}}_2 + \ldots + c_p\underline{\boldsymbol{b}}_p.$$

proof:

<u>Definition</u> (Each basis gives us a coordinate system). Let V be a vector space, and let $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_p\}$ be a basis for V. For each $\underline{\boldsymbol{v}} \in V$ we say that *the coordinates of* $\underline{\boldsymbol{v}}$ *with respect to* β are $c_1, c_2, \ldots c_p$ if

$$\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{b}}_1 + c_2\underline{\boldsymbol{b}}_2 + \ldots + c_p\underline{\boldsymbol{b}}_p.$$

And, we write the vector of the coordinates of $\underline{\boldsymbol{v}}$ with respect to β as:

$$[\underline{\boldsymbol{v}}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ : \\ c_p \end{bmatrix} \in \mathbb{R}^p .$$

<u>Example</u>: For the vector space

$$P_3 = \left\{ p(t) = a_0 + a_1\, t + a_2\, t^2 + a_3\, t^3 \textit{ such that } a_0, a_1, a_2\, , a_3 \in \mathbb{R} \right\}$$

we've checked that

$$\beta = \{\ 1, t, t^2, t^3\ \}$$

is a basis. So the coordinate vector of

$$p(t) = 3 - 4\,t^2 + t^3$$

with respect to β is

$$[p]_\beta = \begin{bmatrix} 3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

And, if $q \in P_3$, with

$$[q]_\beta = \begin{bmatrix} -2 \\ 1 \\ 7 \\ 0 \end{bmatrix}$$

then

$$q(t) = -2 + t + 7t^2.$$

It turns out that we can understand pretty much any vector space question about P_3 by interpreting the question in terms of the coordinates with respect to β, which lets us work in $\mathbb{R}^4$ in lieu of P_3. That's what coordinates with respect to a basis are good for, when you're working with a non-standard vector space.

Exercise 1) Let

$$\beta = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\} = \{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}\}$$

be a non-standard basis of $\mathbb{R}^2$.

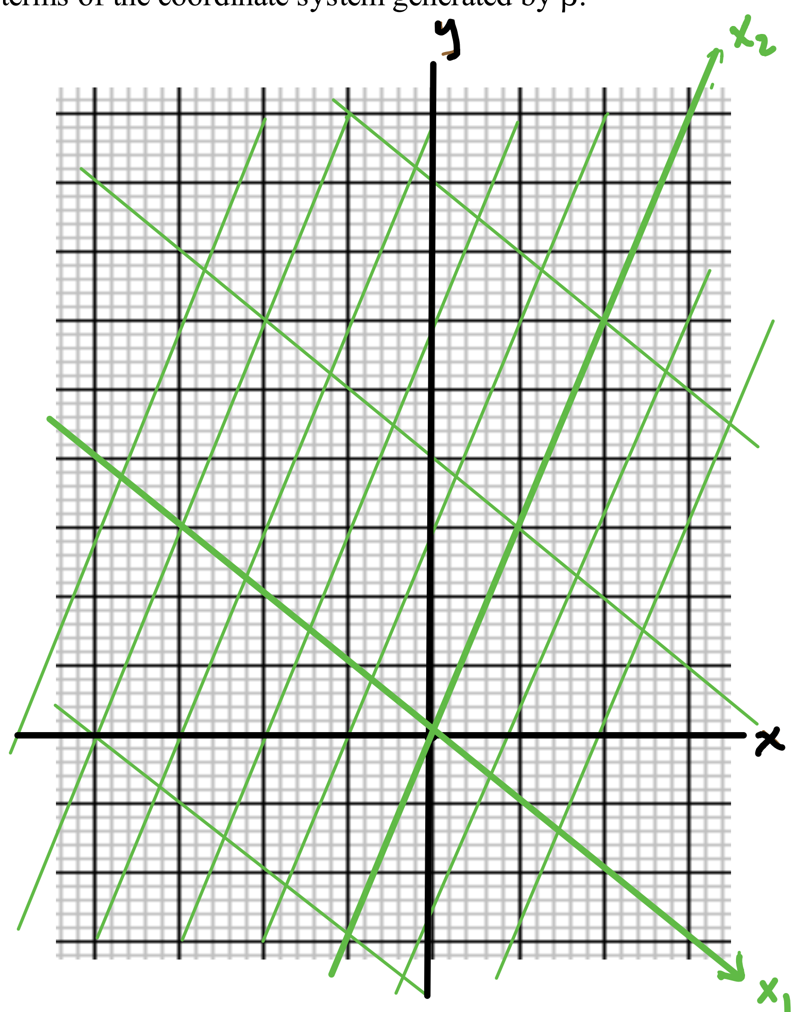
1a) Suppose $\underline{\boldsymbol{x}}$ is a vector in $\mathbb{R}^2$, and

$$[\underline{\boldsymbol{x}}]_\beta = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Find the standard coordinates for $\underline{\boldsymbol{x}}$, i.e. its coordinates with respect to the standard basis $E = \{\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2\}$.

1b) Find the β-coordinates for the vector $\underline{\boldsymbol{b}} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$. (The math may seem familiar.)

1c) Interpret your work in 1ab geometrically, in terms of the coordinate system generated by β.



Theorem Let V, W be vector spaces, and $T : V \to W$ a linear transformation. If T is $1 - 1$ and onto, then the inverse function T^{-1} is also a linear transformation, $T^{-1} : W \to V$. In this case, we call T an *isomorphism*.

proof: We have to check that for all $\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}} \in W$ and all $c \in \mathbb{R}$,

$$T^{-1}(\underline{\boldsymbol{u}} + \underline{\boldsymbol{w}}) = T^{-1}(\underline{\boldsymbol{u}}) + T^{-1}(\underline{\boldsymbol{w}})$$

$$T^{-1}(c\,\underline{\boldsymbol{u}}) = c\,T^{-1}(\underline{\boldsymbol{u}}).$$

Theorem Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$. Then the coordinate transform $T : V \to \mathbb{R}^n$ defined by

$$T(\underline{\boldsymbol{v}}) = [\underline{\boldsymbol{v}}]_\beta$$

is linear, and it is an isomorphism.

Exercise 2: Use coordinates with respect to the basis $\{1, t, t^2\}$, to check whether or not the set of polynomials $\{p_1(t), p_2(t), p_3(t)\}$ is a basis for $\boldsymbol{P}_2$, where

$$p_1(t) = 1 + t^2$$
$$p_2(t) = 2 + 3t + t^2$$
$$p_3(t) = -3t + t^2 .$$

Exercise 3 Generalize the example of Exercise 1: Suppose $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ \ldots \underline{\boldsymbol{b}}_n\}$ is a non-standard basis of $\mathbb{R}^n$. And let $E = \{\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2, \ \ldots \underline{\boldsymbol{e}}_n\}$ be the standard basis of $\mathbb{R}^n$. For $\underline{\boldsymbol{x}} \in \mathbb{R}^n$, how do you convert between $\underline{\boldsymbol{x}} = [\underline{\boldsymbol{x}}]_E$, and $[\underline{\boldsymbol{x}}]_\beta$, and vise-verse?

Wed Feb 28

- 4.5 dimension of a vector space, and related facts about span and linear independence.

Announcements:

Warm-up Exercise:

There is a circle of ideas related to linear independence, span, and bases for vector spaces, which it is good to try and understand carefully. That's what we'll do today. These ideas generalize (and use) ideas we've already explored more concretely, and facts we already know to be true for the vector spaces $\mathbb{R}^n$.

<u>Theorem 1</u> (constructing a basis from a spanning set): Let V be a vector space of dimension at least one, and let $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\} = V$.
Then a subset of the spanning set is a basis for V. (We followed a procedure like this to extract bases for *Col A*.)

<u>Theorem 2</u> Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$. Then any set in V containing more than n elements must be linearly dependent. (We used reduced row echelon form to understand this in $\mathbb{R}^n$.)

Theorem 3 Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ \dots \underline{\boldsymbol{b}}_n\}$. Let $\alpha = \{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ \dots \underline{\boldsymbol{a}}_p\}$ be a set of independent vectors that don't span V. Then $p < n$, and additional vectors can be added to the set α to create a basis $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ \dots \underline{\boldsymbol{a}}_p, \dots \underline{\boldsymbol{a}}_n\}$ (We followed a procedure like this when we figured out all the subspaces of $\mathbb{R}^3$.)

Theorem 4 Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ \dots \underline{\boldsymbol{b}}_n\}$. Then no set $\alpha = \{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ \dots \underline{\boldsymbol{a}}_p\}$ with $p < n$ vectors can span V. (We know this for $\mathbb{R}^n$.)

Theorem 5 Let Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ \dots \ \underline{\boldsymbol{b}}_n\}$. Then every basis for V has exactly n vectors. Furthermore, if $\alpha = \{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ \dots \ \underline{\boldsymbol{a}}_n\}$ is another collection of exactly n vectors in V, and if $span\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ \dots \ \underline{\boldsymbol{a}}_n\} = V$, then the set α is automatically linearly independent and a basis. Conversely, if the set $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ \dots \ \underline{\boldsymbol{a}}_n\}$ is linearly independent, then $span\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ \dots \ \underline{\boldsymbol{a}}_n\} = V$ is guaranteed, and α is a basis. (We know all these facts for $\mathbb{R}^n$ from reduced row echelon form considerations.)

Corollary Let Let V be a vector space of dimension n. Then the subspaces of V have dimensions $0, 1, 2, \dots n - 1, n$. (We know this for $\mathbb{R}^n$.)

Fri Mar 2

- 4.6 The four subspaces associated with a matrix. the rank of a matrix.

Announcements:

Warm-up Exercise:

Let A be an $m \times n$ matrix. There are four subspaces associated with A. To keep them straight, keep in mind the associated linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^m \text{ given by } T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}.$$

And, as usual, we can express A in terms of its columns, $A = \begin{bmatrix} \underline{\boldsymbol{a}}_1 & \underline{\boldsymbol{a}}_2 & \ldots & \underline{\boldsymbol{a}}_n \end{bmatrix}$. Then the two subspaces we know well are

$$Col\, A = span\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ \ldots\ \underline{\boldsymbol{a}}_n\} \subseteq \mathbb{R}^m$$

$$Nul\, A = \{\underline{\mathbf{x}} \in \mathbb{R}^n : A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}\} \subseteq \mathbb{R}^n.$$

And, in your homework you already figured out the "rank + nullity" theorem, that

$$dim\,(Col\, A) + dim\,(Nul\, A) = n.$$

The reason for this is that if p is the number of pivots in the reduced row echelon form of A, then

$$\begin{aligned} dim\,(Col\, A) &= p \\ dim\,(Nul\, A) &= n - p. \end{aligned}$$

The number of pivots, i.e. $dim\,(Col\, A)$ is called the *rank* of the matrix A. What are the other two subspaces and why do we care? Well,

- First, recall the geometry fact that the dot product of two vectors in $\mathbb{R}^n$ is zero if and only if the vectors are perpendicular, i.e.

$$\underline{\boldsymbol{u}} \cdot \underline{\boldsymbol{v}} = 0 \quad \text{if and only if } \underline{\boldsymbol{u}} \perp \underline{\boldsymbol{v}}.$$

(Well, we really only know this in $\mathbb{R}^2$ or $\mathbb{R}^3$ so far, from multivariable Calculus class. But it's true for all $\mathbb{R}^n$, as we'll see in Chapter 6.) So for a vector $\underline{\boldsymbol{x}} \in Nul\,A$ we can interpret the equation

$$A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$$

as saying that $\underline{\boldsymbol{x}}$ is perpendicular to every row of A. Because the dot product distributes over addition, we see that each $\underline{\boldsymbol{x}} \in Nul\,A$ is perpendicular to every linear combination of the rows of A. This motivates the next subspace associated with A, namely the rowspace. In other words, if we express A in terms of its rows,

$$A = \begin{bmatrix} \underline{\boldsymbol{R}}_1 \\ \underline{\boldsymbol{R}}_2 \\ : \\ \underline{\boldsymbol{R}}_m \end{bmatrix}$$

then

$$Row\,A := span\{\underline{\boldsymbol{R}}_1, \underline{\boldsymbol{R}}_2, \ldots \underline{\boldsymbol{R}}_m\} \subseteq \mathbb{R}^n.$$

Notice, that as we do elementary operations on the rows of A we don't change their span, so we get a great basis for $Row\,A$ by using the non-zero rows of $rref(A)$. So, the dimension of $Row(A)$ is p, the number of pivots in the reduced matrix. So in the domain $\mathbb{R}^n$, we have this picture:

$$\begin{aligned} dim(Nul\,A) &= n - p \\ dim(Row\,A) &= p \\ Nul\,A &\perp Row\,A. \end{aligned}$$

The final subspace lives in the codomain $\mathbb{R}^m$, along with $Col\,A$. Well, $Col\,A = Row\,A^T$. And so $Nul\,A^T$ is the final subspace. It's perpendicular to $Col(A)$ and will have dimension $m - p$. So in the codomain $\mathbb{R}^m$ we have the picture

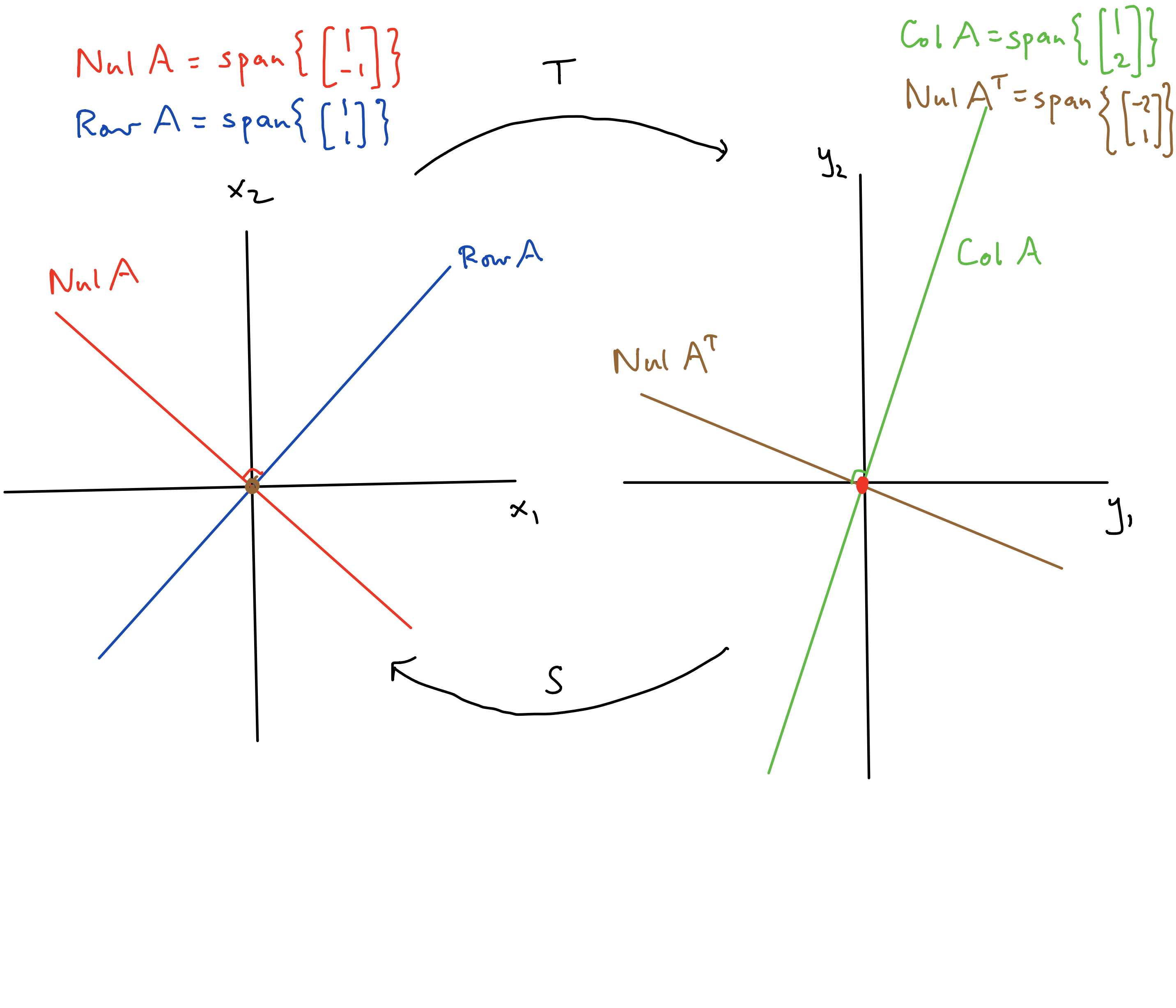
$$\begin{aligned} dim(Col\,A) &= p \\ dim(Nul\,A^T) &= m - p \\ Nul\,A^T &\perp Row\,A^T = Col\,A \end{aligned}$$

small example.

$$T:\mathbb{R}^2 \to \mathbb{R}^2$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$S\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$



I'll bring an example to play with on Friday! (The book has one, in the mean time.) Here's a schematic of what's going on, stolen from the internet.