Math 2270-004 Week 7 notes

We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 4.1-.4.3.

Tues Feb 20

- 4.1 Vector spaces and subspaces.

└ collections of objects that you can add & scalar multiply ...

Announcements: Some of 4.1 HW: 1, 3, 5, 7, 9, 13, 19, 21, 23, 31

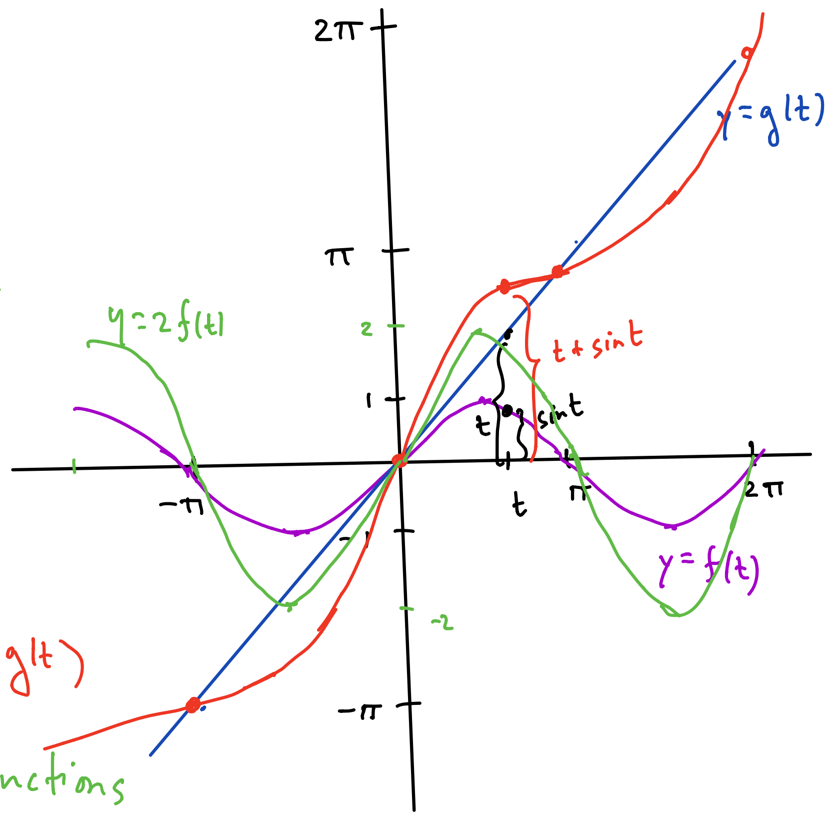
Warm-up Exercise: 'til 12:57

Let $f(t) = \sin t$

$g(t) = t$

Graph $f+g$, $2f$

on graph $y = f(t) + g(t)$ vertical displacement @ t is the sum of the vertical displacements of $y = f(t)$, $y = g(t)$



You can add and scalar multiply functions

We dealt with vector equations - linear combination equations - a lot, in Chapters 1-2. There are other objects, for example functions, that one can add and scalar mutiply. The useful framework that includes these other examples is the abstract concept of a *vector space*. There is a body of results that only depends on the axioms below, and not on the particular vector space in which one is working. So rather than redo those results in different-seeming contexts, we understand them once abstractly, in order to apply them in many places. And, we can visualize the abstractions be relying on our solid foundation in $\mathbb{R}^n$.

<u>Definition</u> A *vector space* is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *scalar multiplication,* so that the ten axioms listed below hold. These axioms must hold for all vectors $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}}$ in V, and for all scalars $c, d \in \mathbb{R}$.

<u>for example 4</u>

1. The sum of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$, denoted by $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}$, is (also) in V (closure under addition.)

$f, g \in V$, then we said $(f+g)(t) := f(t)+g(t)$

2. $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} = \underline{\boldsymbol{v}} + \underline{\boldsymbol{u}}$ (commutative property of addition)

$f+g \overset{?}{=} g+f$ at t: $f(t)+g(t) \overset{?}{=} g(t)+f(t)$

3. $(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) + \underline{\boldsymbol{w}} = \underline{\boldsymbol{u}} + (\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}})$ (associative property of addition)

$(f+g)+h \overset{?}{=} f+(g+h)$ at t:

$LHS = (f+g)(t) + h(t)$

4. There is a **zero** vector $\underline{\mathbf{0}}$ in V so that $\underline{\boldsymbol{u}} + \underline{\mathbf{0}} = \underline{\boldsymbol{u}}$. (additive identity)

$\vec{0}$: is the function $f(t)=0$ for all t

$= (f(t)+g(t)) + h(t)$

$= RHS$.

5. For each $\underline{\boldsymbol{u}} \in V$ there is a vector $-\underline{\boldsymbol{u}} \in V$ so that $\underline{\boldsymbol{u}} + (-\underline{\boldsymbol{u}}) = \underline{\mathbf{0}}$. (additive inverses)

the function $-f$: $(-f)(t) = -f(t)$

6. The scalar multiple of $\underline{\boldsymbol{u}}$ by c, denoted by $c\,\underline{\boldsymbol{u}}$ is (also) in V. (closure under scalar multiplication)

⋮

7. $c(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = c\,\underline{\boldsymbol{u}} + c\,\underline{\boldsymbol{v}}$ (scalar multiplication distributes over vector addition)

8. $(c + d)\,\underline{\boldsymbol{u}} = c\,\underline{\boldsymbol{u}} + d\,\underline{\boldsymbol{u}}$. (scalar addition distributes over scalar multiplication of vectors)

9. $c\,(d\,\underline{\boldsymbol{u}}) = (c\,d)\,\underline{\boldsymbol{u}}$ (associative property of scalar multiplication)

10. $1\,\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$ (multiplicative identity)

for function vector spaces the axioms hold because once we evaluate at "t" they just reduce to properties of real number addition.

The following three algebra rules follow from the first 10, and are also useful:

11) $0\,\underline{\boldsymbol{u}} = \underline{\mathbf{0}}$.

12) $c\,\underline{\mathbf{0}} = \underline{\mathbf{0}}$.

13) $-\underline{\boldsymbol{u}} = (-1)\,\underline{\boldsymbol{u}}$.

Example 1 $\mathbb{R}^n$, $n \geq 1$ a positive integer, with vector addition and scalar multiplication defined component-wise, and with the zero vector being the vector which has every entry equal to zero. Then the properties above just reduce to properties of real number algebra.

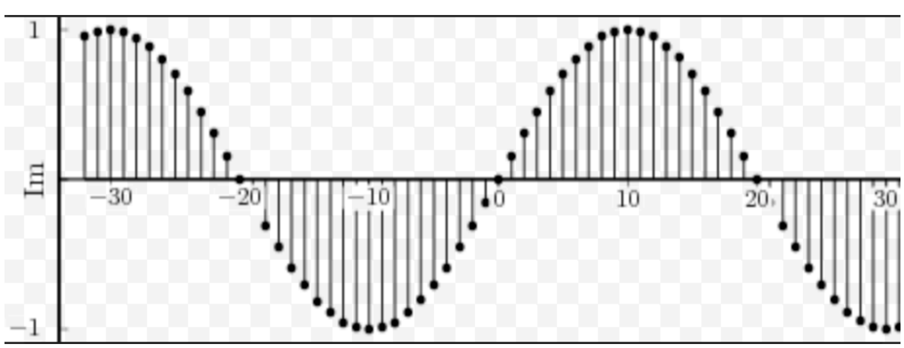
Example 2 The space of $m \times n$ matrices (with m, n fixed), with matrix addition and scalar multiplication defined component-wise, and with the zero vector being the matrix which has every entry equal to zero.

Example 3 The set S of doubly-infinite sequences of numbers. (Think of this as discrete time *signals.*) So an element of S can be written as

$$\{y_k\} = (\; .... \; y_{-2}\,, y_{-1}\,, y_0\,, y_1\,, y_2, y_3\,, \ldots)$$

If $\{z_k\}$ is another element of , then the sum $\{y_k\} + \{z_k\}$ is the sequence $\{y_k + z_k\}$, and the scalar multiple $c\{y_k\}$ is the sequence $\{\, c\, y_k\}$. Checking the vector space axioms is the same as it would be $\mathbb{R}^n$, since addition and scalar multiplication are done entry by entry. (There's just infinitely many entries.)

One can visualize such a discrete time signal in terms of its graph over over its integer "domain". How could you visualize "vector" addition and scalar multiplication in this case?



Example 4 Let V be the set of all real-value functions defined on a domain D on the real line. (In practice D is usually an interval or the entire real line.) If f and g are in V, then we define their sum and scalar multiples "component-wise" too, where each $t \in$ D gives a component:

$$(f+g)(t) := f(t) + g(t)$$

$$(c f)\ (t) := c f(t).$$

So for example, the domain is the entire real line and if f is defined by the rule $f(t) = 1 + 2\sin(3\,t)$ and g is defined by the rule $g(t) = 3\,e^t$ then the function $f + g$ is defined by the rule

$f(t) + g(t)$

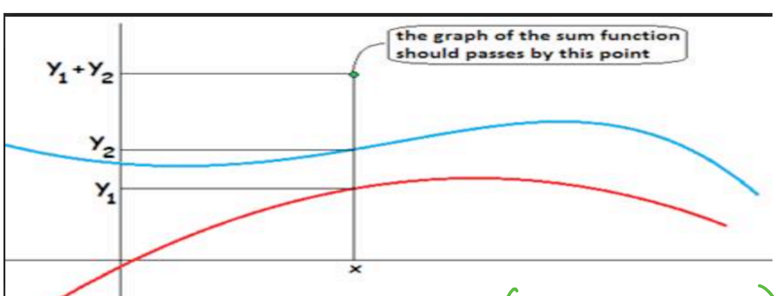
$$(f+g)(t) = 1 + 2\sin(3\,t) + 3\,e^t$$

and $7\,f$ is the function defined by the rule

$$(7\,f)\ (t) = 7(1 + 2\sin(3\,t)) = 7 + 14\sin(3\,t).$$

Why do the vector space axioms hold? What is the zero vector in this vector space? What is the additive inverse function?

One can visualize functions in terms of their graphs - just like for the discrete time signals - and then the graphs of the sum of two functions or of a scalar multiple of one function, are contructed as you'd expect:



(see warmup.)

think sub-vector-space

Definition: A *subspace* of a vector space V is a subset H of V which is itself a vector space with respect to the addition and scalar multiplication in V. As soon as one verifies a), b), c) below for H, it will be a subspace, because H will "inherit" the other axioms just by being contained in V.

a) The zero vector of V is in H (property 4)

b) H is closed under vector addition, i.e. for each $\underline{\boldsymbol{u}} \in H, \underline{\boldsymbol{v}} \in H$ then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in H$. (property 1)

c) H is closed under scalar multiplication, i.e for each $\underline{\boldsymbol{u}} \in H, c \in \mathbb{R}$, then also $c\,\underline{\boldsymbol{u}} \in H$. (property 6)

(c) ⇒ (a)
$0\vec{u} = \vec{0}$

Just to double check that the other properties get inherited:

Definition A *vector space* is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *scalar multiplication,* so that the ten axioms listed below hold. These axioms must hold for all vectors $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}}$ in V, and for all scalars $c, d \in \mathbb{R}$.

(b) 1. The sum of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$, denoted by $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}$, is (also) in V (closure under addition.)

2. $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} = \underline{\boldsymbol{v}} + \underline{\boldsymbol{u}}$ (commutative property of addition)

3. $(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) + \underline{\boldsymbol{w}} = \underline{\boldsymbol{u}} + (\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}})$ (associative property of addition)

(a) 4. There is a **zero** vector $\underline{\mathbf{0}}$ in V so that $\underline{\boldsymbol{u}} + \underline{\mathbf{0}} = \underline{\boldsymbol{u}}$. (additive identity)

13 & (c) 5. For each $\underline{\boldsymbol{u}} \in V$ there is a vector $-\underline{\boldsymbol{u}} \in V$ so that $\underline{\boldsymbol{u}} + (-\underline{\boldsymbol{u}}) = \underline{\mathbf{0}}$. (additive inverses)

(c) 6. The scalar multiple of $\vec{u}$ by c, denoted by $c\,\vec{u}$ is (also) in V. (closure under scalar multiplication)

7. $c(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = c\,\underline{\boldsymbol{u}} + c\,\underline{\boldsymbol{v}}$ (scalar multiplication distributes over vector addition)

8. $(c + d)\,\underline{\boldsymbol{u}} = c\,\underline{\boldsymbol{u}} + d\,\underline{\boldsymbol{u}}$. (scalar multiplication distributes over scalar addition)

9. $c\,(d\,\underline{\boldsymbol{u}}) = (c\,d)\,\underline{\boldsymbol{u}}$ (associative property of scalar multiplication)

10. $1\,\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$ (multiplicative identity)

The following three algebra rules follow from the first 10, and are also useful:

11) $0\,\underline{\boldsymbol{u}} = \vec{0}$ $\vec{0}$

12) $c\,\underline{\mathbf{0}} = \underline{\mathbf{0}}$.

13) $-\underline{\boldsymbol{u}} = (-1)\,\underline{\boldsymbol{u}}$.

Big Exercise: The vector space $\mathbb{R}^n$ has subspaces! But there aren't very many kinds, it turns out. (Even though there are countless kinds of *subsets* of $\mathbb{R}^n$.) Let's find *all* the possible kinds of subspaces of $\mathbb{R}^3$, using our expertise with matrix reduced row echelon form.

All sub (vector) spaces of $\mathbb{R}^3$. From small to large.

Let H be a subspace of $\mathbb{R}^3$

(a) $\vec{0} \in H$

(b) $\vec{u}, \vec{v} \in H \Rightarrow \vec{u}+\vec{v} \in H$

(c) $\vec{u} \in H \Rightarrow c\vec{u} \in H$ for all $c \in \mathbb{R}$.

- $\vec{0} \in H$ by (a)
- $\longrightarrow$ H could be $\{\vec{0}\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

if more to H, let $\vec{u} \in H$, $\vec{u} \neq \vec{0}$ $\qquad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

by (c) $\text{span}\{\vec{u}\} = \{c\vec{u} \text{ s.t. } c \in \mathbb{R}\}$ is contained in H

note (a), (b), (c) hold for $\text{span}\{\vec{u}\}$

- $\longrightarrow$ so H could be $\text{span}\{\vec{u}\}$ $=$ line thru $\vec{0}$, with direction vector $\vec{u}$.

If there's more to H, let $\vec{v} \in H$, $\vec{v} \neq c\vec{u}$ for any c

(c) says $\text{span}\{\vec{v}\} \subset H$ ($\subset$: subset)

- $\xrightarrow[H \text{ could be}]{}$ (b) $\Rightarrow$ $\underbrace{\text{span}\{\vec{u},\vec{v}\}}_{\text{is a subspace:}} \subset H$ $\quad$ ($c_1\vec{u} \in H$, $c_2\vec{v} \in H \Rightarrow c_1\vec{u}+c_2\vec{v} \in H$

(a) $\vec{0} = 0\vec{u} \in H$

(b) $(c_1\vec{u}+c_2\vec{v}) + (d_1\vec{u}+d_2\vec{v}) = (c_1+d_1)\vec{u}+(c_2+d_2)\vec{v} \in H$

(c) $c(c_1\vec{u}+c_2\vec{v}) = cc_1\vec{u}+cc_2\vec{v} \in H$

plane thru $\vec{0}$.

$$\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ says } \vec{v} = *\vec{u}}$$

- There could be more than just the $\text{span}\{\vec{u},\vec{v}\}$ in H.

So, there is $\vec{w} \in H$, $\vec{w} \notin \text{span}\{\vec{u},\vec{v}\}$

Since H is closed under $+$ & scalar multiplication

deduce $\text{span}\{\vec{u},\vec{v},\vec{w}\}$ is in H.

$\text{span}\{\vec{u},\vec{v},\vec{w}\} = \mathbb{R}^3$

because

$$\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to \cancel{\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}}$$

would say that

$\vec{w} = a\vec{u} + b\vec{v}$

This can't happen because we assumed $\vec{w}$ was NOT in span $\{\vec{u}, \vec{v}\}$.

since $[\vec{u}\ \vec{v}\ \vec{w}]$ reduces to I.

we can always (uniquely solve)

$$[\vec{u}\ \vec{v}\ \vec{w}]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{b} \quad \text{for } \vec{x} \quad (\text{for every } \vec{b} \in \mathbb{R}^3)$$

i.e. $\text{span}\{\vec{u}, \vec{v}, \vec{w}\} = \mathbb{R}^3$

Wed Feb 21

- 4.1-4.2 Vector spaces and subspaces; null spaces, column spaces, and the connections to linear transformations

Announcements:
- Homework for next week includes

4.1 1, 3, 5, 7, 9, 13, 19, 21, 23, 31

4.2 1, 3, 7, 9, 15, 17, 21, 25, 27, 31, 33

- Quiz today

Warm-up Exercise: 'til 12:57

The matrix
$$\begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ \end{array} \quad\quad \begin{array}{ccc} \vec{r}_1 & \vec{r}_2 & \vec{r}_3 \end{array}$$
$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & -7 \\ 3 & 1 & 8 \end{bmatrix} \text{ reduces to } \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

express $\begin{bmatrix} 1 \\ -7 \\ 8 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 \\ -7 \\ 8 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

review

column depencies of a matrix correspond to solutions $\vec{x}$ to $A\vec{x} = \vec{0}$

these stay the same as you reduce the matrix:

$$\vec{r}_3 = 3\vec{r}_1 - \vec{r}_2 \qquad \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

So $\vec{a}_3 = 3\vec{a}_1 - \vec{a}_2$

(or $\vec{0} = 3\vec{a}_1 - \vec{a}_2 - \vec{a}_3$

$$A\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} = \vec{0}$$

We've been discussing the abstract notions of *vector spaces* and *subspaces,* with some specific examples to help us with our intuition. Today we continue that discussion. We'll continue to use exactly the same language we used in Chapters 1-2 .... except now it's for general vector spaces:

Let V be a *vector space* (Do you recall that definition, at least roughly speaking?)

<u>Definition:</u> If we have a collection of p vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$ in V, then any vector $\underline{\boldsymbol{v}} \in V$ that can be expressed as a sum of scalar multiples of these vectors is called a *linear combination* of them. In other words, if we can write

$$\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_p\underline{\boldsymbol{v}}_p ,$$

then $\underline{\boldsymbol{v}}$ is a *linear combination* of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the *linear combination coefficients* or *weights.*

<u>Definition</u> The *span* of a collection of vectors, written as $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$, is the collection of all linear combinations of those vectors.

<u>Definition:</u>
<u>a)</u> An indexed set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$ in V is said to be *linearly independent* if no one of the vectors is a linear combination of (some) of the other vectors. The concise way to say this is that the only way $\underline{\mathbf{0}}$ can be expressed as a linear combination of these vectors,

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_p\underline{\boldsymbol{v}}_p = \underline{\mathbf{0}} ,$$

is for all of the weights $c_1 = c_2 = \ldots = c_p = 0$.

<u>b)</u> An indexed set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$ is said to be *linearly dependent* if at least one of these vectors is a linear combination of (some) of the other vectors. The concise way to say this is that there *is* some way to write $\underline{\mathbf{0}}$ as a linear combination of these vectors

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_p\underline{\boldsymbol{v}}_p = \underline{\mathbf{0}}$$

where *not all* of the $c_j = 0$. (We call such an equation a *linear dependency.* Note that if we have any such linear dependency, then any $\underline{\boldsymbol{v}}_j$ with $c_j \neq 0$ is a linear combination of the remaining $\underline{\boldsymbol{v}}_k$ with $k \neq j$. We say that such a $\underline{\boldsymbol{v}}_j$ is *linearly dependent* on the remaining $\underline{\boldsymbol{v}}_k$.)

And from yesterday,

Definition: A *subspace* of a vector space V is a subset H of V which is itself a vector space with respect to the addition and scalar multiplication in V. As soon as one verifies a), b), c) below for H, it will be a subspace.

a) The zero vector of V is in H

b) H is closed under vector addition, i.e. for each $\underline{\boldsymbol{u}} \in H$, $\underline{\boldsymbol{v}} \in H$ then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in H$.

c) H is closed under scalar multiplication, i.e for each $\underline{\boldsymbol{u}} \in H$, $c \in \mathbb{R}$, then also $c\,\underline{\boldsymbol{u}} \in H$.

Theorem (spans are subspaces) Let V be a vector space, and let $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots\ \underline{\boldsymbol{v}}_n\}$ be a set of vectors in V.
Then $H = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots\ \underline{\boldsymbol{v}}_n\}$ is a subspace of V.
proof: We need to check that for $H = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots\ \underline{\boldsymbol{v}}_n\}$

a) The zero vector of V is in H

$\vec{0} = 0\vec{v}_1 \in H$ (11 in vector space axioms)

b) H is closed under vector addition, i.e. for each $\underline{\boldsymbol{u}} \in H$, $\underline{\boldsymbol{v}} \in H$ then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in H$.

Let $\vec{u}, \vec{v} \in span\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$

So $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$

$\vec{v} = d_1\vec{v}_1 + d_2\vec{v}_2 + \ldots + d_n\vec{v}_n$

don't need, because addition is associative

so $\vec{u}+\vec{v} = (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) + (d_1\vec{v}_1 + d_2\vec{v}_2 + \ldots + d_n\vec{v}_n)$

$= (c_1\vec{v}_1 + d_1\vec{v}_1) + (c_2\vec{v}_2 + d_2\vec{v}_2) + \ldots + (c_n\vec{v}_n + d_n\vec{v}_n)$ addition is commutative

c) H is closed under scalar multiplication, i.e for each $\underline{\boldsymbol{u}} \in H$, $c \in \mathbb{R}$, then also $c\,\underline{\boldsymbol{u}} \in H$.

$= (c_1 + d_1)\vec{v}_1 + (c_2 + d_2)\vec{v}_2 + \ldots + (c_n + d_n)\vec{v}_n$

scalar addition distributes over scalar mult.

So $\vec{u}+\vec{v} \in H$

$c(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n) = cc_1\vec{v}_1 + cc_2\vec{v}_2 + \ldots + cc_n\vec{v}_n$

property (7)

so $c\vec{u} \in H$

Remark Using minimal spanning sets was how we were able to characterize all possible subspace of $\mathbb{R}^3$ yesterday (or today, if we didn't finish on Tuesday). Can you characterize all possible subsets of $\mathbb{R}^n$ in this way?

Example: Let P_n be the space of polynomials of degree at most n,

$$P_n = \{p(t) = a_0 + a_1 t + a_2 t^2 + ... + a_n t^n \text{ such that } a_0, a_1, ... a_n \in \mathbb{R}\}$$

Note that P_n is the span of the $(n+1)$ functions

$$p_0(t) = 1, p_1(t) = t, p_2(t) = t^2, ... \; p_n(t) = t^n.$$

Although we often consider P_n as a vector space on its own, we can also consider it to be a subspace of the much larger vector space V of all functions from $\mathbb{R}$ to $\mathbb{R}$.

Exercise 1 abbreviating the functions by their formulas, we have

$$P_3 = span\{\, 1, t, t^2, t^3 \,\}.$$

Are the functions in the set $\{\, 1, t, t^2, t^3 \,\}$ linearly independent or linearly dependent?.

assuming domain is $\mathbb{R}$ for these functions.

dependency equation

$$c_1 \cdot 1 + c_2 t + c_3 t^2 + c_4 t^3 \equiv 0 \qquad (\forall t)$$

by fund. thm of algebra a cubic (or lower degree) polynomial as at most 3 roots. (unless it's the zero polynomial)

since $\mathbb{R}$ has more than 3 points we deduce we have the zero polynomial, i.e. $c_1 = c_2 = c_3 = c_4 = 0$

better: If $c_1 + c_2 t + c_3 t^2 + c_4 t^3 \equiv 0$ $\Rightarrow c_1 = 0$.

$\frac{d}{dt}$: $c_2 + c_3 2t + c_4 3t^2 \equiv \frac{d}{dt} 0 \equiv 0$ $\Rightarrow c_2 = 0$

$\frac{d}{dt}$: $2c_3 + 6c_4 t \equiv 0$ $\Rightarrow c_3 = 0$

$\frac{d}{dt}$: $6c_4 = 0$ $\longrightarrow c_4 = 0$

4.2 Null spaces, column spaces, and linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Definition Let A be an $m \times n$ matrix, expressed in column form as $A = [\underline{\boldsymbol{a}}_1\ \underline{\boldsymbol{a}}_2\ \underline{\boldsymbol{a}}_3\ \dots\ \underline{\boldsymbol{a}}_n]$ The *column space* of A, written as $Col\,A$, is the span of the columns:

$$Col\,A = span\{\underline{\boldsymbol{a}}_1\ \underline{\boldsymbol{a}}_2\ \underline{\boldsymbol{a}}_3\ \dots\ \underline{\boldsymbol{a}}_n\}.$$

Equivalently, since

$$A\,\underline{\mathbf{x}} = x_1\,\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \dots + x_n\,\underline{\boldsymbol{a}}_n$$

we see that $Col\,A$ is also the range of the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$, i.e

$$Col\,A = \{\underline{\boldsymbol{b}} \in \mathbb{R}^m \text{ such that } \underline{\boldsymbol{b}} = A\,\underline{\mathbf{x}} \text{ for some } \underline{\mathbf{x}} \in \mathbb{R}^n\}.$$

Theorem By the "spans are subspaces" theorem, $Col(A)$ is always a subspace of $\mathbb{R}^m$.

Exercise 2a) Consider

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix}.$$

By the Theorem, $col(A)$ is a subspace of $\mathbb{R}^3$. Which is it: $\{\underline{\mathbf{0}}\}$, a line thru the origin, a plane thru the origin, or all of $\mathbb{R}^3$. Hint:

$$\begin{array}{ccccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \end{array} \qquad\qquad \begin{array}{ccccc} \vec{r}_1 & \vec{r}_2 & \vec{r}_3 & \vec{r}_4 & \vec{r}_5 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix} \text{ reduces to } \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\vec{a}_1 \neq \vec{0}$
$\vec{a}_2 = -2\vec{a}_1$
$\vec{a}_3$ ind of $\vec{a}_1$
$\vec{a}_4 = -\vec{a}_1 + \vec{a}_3$
$\vec{a}_5 = \vec{a}_1 + \vec{a}_3$

$\Longleftarrow$

$\vec{r}_1 \neq \vec{0}$
$\vec{r}_2 = -2\vec{r}_1$
$\vec{r}_3$ ind. of $\vec{r}_1$
$\vec{r}_4 = -\vec{r}_1 + \vec{r}_3$
$\vec{r}_5 = \vec{r}_1 + \vec{r}_3$

2b) Is there a more efficient way to express $Col\,A$ as a span that doesn't require all five column vectors?

YES.

$$c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 + c_4\vec{a}_4 + c_5\vec{a}_5$$
$$= c_1\vec{a}_1 + c_2(-2\vec{a}_1) + c_3\vec{a}_3 + c_4(-\vec{a}_1 + \vec{a}_3) + c_5(\vec{a}_1 + \vec{a}_3)$$
$$= d_1\vec{a}_1 + d_3\vec{a}_3$$

$$Col\,A = span\{\vec{a}_1, \vec{a}_2, \dots \vec{a}_5\} = span\{\vec{a}_1, \vec{a}_3\}$$

so, col A was just a plane thru origin

Definition: If a set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ in a vector space V is linearly independent and also spans V, then the collection is called a *basis* for V.

Exercise 3 Exhibit a basis for *col A* in Exercise 2.

$$\{\vec{a}_1, \vec{a}_3\} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}.$$

Exercise 4 Exhibit a basis for P_3 in Exercise 1

$$\overset{\shortparallel}{\{a_0 + a_1 t + a_2 t^2 + a_3 t^3\}} = \text{span}\{1, t, t^2, t^3\}$$

and we showed $\{1, t, t^2, t^3\}$ are lin-ind., so a basis

Fri Feb 23

- 4.2 - 4.3 nullspaces and column spaces; kernel and range of linear transformations as subspaces.

Linearly independent sets and bases for vector spaces.

Announcements:
- start in Wed. notes
- fft : (good lead in to HW).

Warm-up Exercise: The space of 2×2 matrices $M_{2\times2} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R} \right\}$ is a vector space. Show that

'til 12:57

$$H = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, a, b \in \mathbb{R} \right\} \text{ is a subspace of } M_{2\times2}$$

Hint: you could show H contains the zero matrix and is closed under matrix addition & scalar multiplication, but it's quicker to show it's the span of two matrices.

Review
- vector space
- subspaces (sub vector spaces) H
  - a) $\vec{0} \in H$
  - b) $\vec{u}, \vec{v} \in H \Rightarrow \vec{u}+\vec{v} \in H$
  - c) $\vec{u} \in H, c \in \mathbb{R} \Rightarrow c\vec{u} \in H$
- if $H = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ then H is a subspace.

1) verify a), b), c)

a) $\vec{0} \in H$: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in H$ $a=b=0$

b) $\left.\begin{matrix} a_{11} = a_{22} = a \\ a_{21} = 0 \end{matrix}\right\}$ condition to be in H

$\vec{u} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$, $\vec{v} = \begin{bmatrix} c & d \\ 0 & c \end{bmatrix}$

$\vec{u}+\vec{v} = \begin{bmatrix} a+c & b+d \\ 0 & a+c \end{bmatrix} \in H$

$a_{11} = a_{22}$
$a_{21} = 0$

c) $\vec{u} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

$e\vec{u} = \begin{bmatrix} ea & eb \\ 0 & ea \end{bmatrix}$ $a_{11} = a_{22}$ ✓ $a_{21} = 0$ so $e\vec{u} \in H$

2) $H = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

from Wed. notes, covered Friday.

4.2 Null spaces, column spaces, and linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Definition Let A be an $m \times n$ matrix, expressed in column form as $A = [\underline{a}_1\ \underline{a}_2\ \underline{a}_3\ \ldots\ \underline{a}_n]$ The *column space* of A, written as $Col\,A$, is the span of the columns:

$$Col\,A = span\{\underline{a}_1\ \underline{a}_2\ \underline{a}_3\ \ldots\ \underline{a}_n\}.$$

Equivalently, since

$$A\,\underline{x} = x_1\,\underline{a}_1 + x_2\,\underline{a}_2 + \ldots + x_n\,\underline{a}_n$$

we see that $Col\,A$ is also the range of the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\underline{x}) = A\,\underline{x}$, i.e

$$Col\,A = \{\underline{b} \in \mathbb{R}^m \text{ such that } \underline{b} = A\,\underline{x} \text{ for some } \underline{x} \in \mathbb{R}^n\}.$$

Theorem By the "spans are subspaces" theorem, $Col(A)$ is always a subspace of $\mathbb{R}^m$.

Exercise 2a) Consider

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix}.$$

By the Theorem, $col(A)$ is a subspace of $\mathbb{R}^3$. Which is it: $\{\underline{0}\}$, a line thru the origin, a plane thru the origin, or all of $\mathbb{R}^3$. Hint:

$$\begin{array}{ccccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \end{array} \qquad\qquad \begin{array}{ccccc} \vec{r}_1 & \vec{r}_2 & \vec{r}_3 & \vec{r}_4 & \vec{r}_5 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix} \text{ reduces to } \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\vec{a}_1 \neq \vec{0}$ | $\vec{r}_1 \neq \vec{0}$
$\vec{a}_2 = -2\vec{a}_1$ | $\vec{r}_2 = -2\vec{r}_1$
$\vec{a}_3$ ind of $\vec{a}_1$ $\Longleftarrow$ $\vec{r}_3$ ind. of $\vec{r}_1$
$\vec{a}_4 = -\vec{a}_1 + \vec{a}_3$ | $\vec{r}_4 = -\vec{r}_1 + \vec{r}_3$
$\vec{a}_5 = \vec{a}_1 + \vec{a}_3$ | $\vec{r}_5 = \vec{r}_1 + \vec{r}_3$

2b) Is there a more efficient way to express $Col\,A$ as a span that doesn't require all five column vectors?

YES.

$$c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 + c_4\vec{a}_4 + c_5\vec{a}_5$$
$$= c_1\vec{a}_1 + c_2(-2\vec{a}_1) + c_3\vec{a}_3 + c_4(-\vec{a}_1 + \vec{a}_3) + c_5(\vec{a}_1 + \vec{a}_3)$$
$$= d_1\vec{a}_1 + d_3\vec{a}_3$$

$$Col\,A = span\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_5\} = span\{\vec{a}_1, \vec{a}_3\}$$

so, col A was just a plane thru origin

from Wed notes, covered Friday

Definition: If a set of vectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n\}$ in a vector space V is linearly independent and also spans V, then the collection is called a *basis* for V.

Exercise 3 Exhibit a basis for *col A* in Exercise 2.

$$\{\vec{a}_1, \vec{a}_3\} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}.$$

Exercise 4 Exhibit a basis for P_3 in Exercise 1

$$P_3 = \{a_0 + a_1 t + a_2 t^2 + a_3 t^3\} = \text{span}\{1, t, t^2, t^3\}$$

and we showed $\{1, t, t^2, t^3\}$ are lin-ind., so a basis

We've seen that one (explicit) way that subspaces arise is as the span of a specified collection of vectors. The primary (implicit) way that subspaces are described is related to the following:

<u>Definition</u>: The *null space* of an $m \times n$ matrix A is the set of $\underline{\mathbf{x}} \in \mathbb{R}^n$ for which $A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$. We denote this set by *Nul A*. Equivalently, in terms of the associated linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$, *Nul A* is the set of points in the domain which are transformed into the zero vector in the codomain.

<u>Theorem</u> Let A be an $m \times n$ matrix. Then *Nul A* is a subspace of $\mathbb{R}^n$.

$H = \{\vec{x} \in \mathbb{R}^n \text{ s.t. } A\vec{x} = \vec{0}\}$

proof: We need to check that for $H = Nul(A)$:

a) The zero vector of V is in H

is $\vec{0} \in H$: $A\vec{0} = \vec{0}$ so $\vec{0} \in H$.

b) H is closed under vector addition, i.e. for each $\underline{\mathbf{u}} \in H$, $\underline{\mathbf{v}} \in H$ *then* $\underline{\mathbf{u}} + \underline{\mathbf{v}} \in H$.

$\Downarrow$

$A\vec{u} = \vec{0}$, $A\vec{v} = \vec{0}$

so $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$

so $\vec{u}+\vec{v} \in H = Nul\,A$

c) H is closed under scalar multiplication, i.e for each $\underline{\mathbf{u}} \in H$, $c \in \mathbb{R}$, *then also* $c\,\underline{\mathbf{u}} \in H$.

$H = Nul\,A$

$\Downarrow$

$A\vec{u} = \vec{0}$.

$\Downarrow$

$A(c\vec{u}) = cA\vec{u} = \vec{0}$

so $c\vec{u} \in Nul\,A$

Exercise 1a) For the same matrix A as in Exercise 2 from Wednesday's notes, express the vectors in $Nul(A)$ explicitly, using the methods of Chapters 1-2. Notice these are vectors in the domain of the associated linear transformation $T : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ given by $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$, so are a subspace of $\mathbb{R}^5$.

$A\vec{x} = \vec{0}$

$$A = \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 1 & 0 \\ -2 & 4 & 0 & 2 & -2 & 0 \\ 3 & -6 & 4 & 1 & 7 & 0 \end{array}\right] \text{ reduces to } \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \leftarrow$$

$\{\vec{x} : A\vec{x} = \vec{0}\}$

implicit description

$x_1 = 2t_2 + t_4 - t_5$
$x_2 = t_2$ free
$x_3 = -t_4 - t_5$
$x_4 = t_4$ free
$x_5 = t_5$ free

explicit description
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = t_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t_5 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

1b) Exhibit a basis for $Nul(A)$.

$$Nul\,A = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

actually a basis

The ideas of nullspace and column space generalize to arbitrary linear transformations between vectors spaces - with slightly more general terminology.

Definition Let V and W be vector spaces. A function $T: V \to W$ is called a *linear transformation* if for each $\underline{\boldsymbol{x}} \in V$ there is a unique vector $T(\underline{\boldsymbol{x}}) \in W$ and so that

(i) $T(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = T(\underline{\boldsymbol{u}}) + T(\underline{\boldsymbol{v}})$ for all $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in V$

(ii) $T(c\,\underline{\boldsymbol{u}}) = c\,T(\underline{\boldsymbol{u}})$ for all $\underline{\boldsymbol{u}} \in V$, $c \in \mathbb{R}$

Definition The *kernel* (or *nullspace)* of T is defined to be $\{\underline{\boldsymbol{u}} \in V:\ T(\underline{\boldsymbol{u}}) = \underline{\mathbf{0}}\}$.

Definition The *range* of T is $\{\underline{\boldsymbol{w}} \in W:\ \underline{\boldsymbol{w}} = T(\underline{\boldsymbol{v}})\ \textit{for some}\ \underline{\boldsymbol{v}} \in V\}$.

Theorem Let $T: V \to W$ be a *linear transformation.* Then the kernel of T is a subspace of V. The range of T is a subspace of W.

Remark: The theorem generalizes our earlier one about *Nul A* and *Col A*, for matrix transformations $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$.

Exercise 2 Let V be the vector space of real-valued functions f defined on an interval $[a, b]$ with the property that they are differentiable and that their derivatives are continuous functions on $[a, b]$. Let W be the vector space $C[a, b]$ of all continous functions on the interval $[a, b]$. Let $\mathrm{D} : V \to W$ be the derivative transformation

$$\mathrm{D}(f) = f' \ .$$

2a) What Calculus differentiation rules tell you that D is a linear transformation?

2b) What subspace is the kernel of D ?

2c) What is the range of D ?