Math 2270-004 Week 6 notes

We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 2.3, 3.1-3.3. In our original course syllabus I had planned to cover 2.4-2.5. These sections are considered optional by the Math Department course coordinator, and in the interests of having enough time for core material that comes later, we'll skip them, at least for now.

2.4 partioned matrices
2.5 matrix product decompositions, as a tool in solving linear systems.
3.1-3.3: determinants

Mon Feb 12

- 2.3 Matrix inverses, the elementary matrix approach
- overview of skipped section 2.5.

Announcements:
- HW posted – due next week
- Quiz this week, on T, W material.
- I'm about 70% finished grading exams (for elementary)

Warm-up Exercise: 'til 12:57

a) Compute the matrix product

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A}$$

b) describe what happened to the rows of A

think rows by row

$$EA = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{11}+a_{21} & 3a_{12}+a_{22} & 3a_{13}+a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

row_1, row_3 stayed the same

$3R_1 + R_2 \longrightarrow R_2$

today: thinking about rref & inverse matrices via "Elementary matrices" that do elementary row ops to matrix A, when we multiply on the left.

We didn't get a chance to discuss the last three statements in the invertible matrix theorem in class, so let's do that before getting to the main part of today's discussion, about the elementary matrix approach to matrix inverses:

<u>The invertible matrix theorem</u> (page 114)

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a) A is an invertible matrix.
b) The reduced row echelon form of A is the $n \times n$ identity matrix.
c) A has n pivot positions
d) The equation $A\,\underline{\boldsymbol{x}} = \underline{\mathbf{0}}$ has only the trivial solution $\underline{\boldsymbol{x}} = \underline{\mathbf{0}}$.
e) The columns of A form a linearly independent set.
f) The linear transformation $T(\underline{\boldsymbol{x}}) := A\,\underline{\boldsymbol{x}}$ is one-one.
g) The equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ has at least one solution for each $\underline{\boldsymbol{b}} \in \mathbb{R}^n$.
h) The columns of A span $\mathbb{R}^n$.
i) The linear transformation $T(\underline{\boldsymbol{x}}) := A\,\underline{\boldsymbol{x}}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j) There is an $n \times n$ matrix C such that $C\,A = I$.

a) $\Rightarrow$ j) Let $C = A^{-1}$

j) $\Rightarrow$ a) : j) $\Rightarrow$ d) $\Rightarrow$ a : if $A\vec{x} = \vec{0}$

$\left.\underbrace{CA}_{I}\vec{x} = C\vec{0} = \vec{0}\right\} \Rightarrow \vec{x} = \vec{0}.$

j, k : half an inverse $\Rightarrow$ all of an inverse.

k) There is an $n \times n$ matrix D such that $A\,\mathrm{D} = I$.

a) $\Rightarrow$ k) Let $D = A^{-1}$

k) $\Rightarrow$ g) $\Rightarrow$ a. to solve $A\vec{x} = \vec{b}$

know $AD = I$

$AD\vec{b} = I\vec{b} = \vec{b}$. Let $\vec{x} = D\vec{b}$!

l) A^T is an invertible matrix.

a) $\Rightarrow$ l). If A^{-1} exists. Call it "B"

so $AB = I$, $BA = I$

take transpose : $(AB)^T = I^T = I$ $\qquad (BA)^T = I^T = I$

$B^T A^T = I$ $\qquad A^T B^T = I$

so $(A^T)^{-1} = B^T = (A^{-1})^T$

l) $\Rightarrow$ a) just use a) $\Rightarrow$ l) on A^T, i.e. if A^T has inverse, C, then $\underbrace{(A^T)^T}_{= A}$ has inverse C^T

2.3: elementary matrix approach to invertible matrices.

<u>Exercise 1)</u> Show that if A, B, C are invertible matrices, then

$$(A\,B)^{-1} = B^{-1}\,A^{-1}.$$
$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

$$(AB)(B^{-1}A^{-1}) = \underbrace{A(\underbrace{BB^{-1}}_{I})}_{A}A^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB)$$
$$= B^{-1}IB$$
$$= B^{-1}B$$
$$= I$$

$$ABC\underbrace{C^{-1}}_{I}B^{-1}A^{-1}$$
$$= ABB^{-1}A^{-1}$$
$$= AA^{-1}$$
$$= I$$

As the examples above show, it is true that

<u>Theorem</u> The product of $n \times n$ invertible matrices is invertible, and the inverse of the product is the product of their inverses in reverse order.

Our algorithm for finding the inverse of a matrix can be reformulated in terms of a product of so-called "elementary" matrices. This product idea will pay off elsewhere. To get started, let's notice an analog of the fact that a matrix times a vector is a linear combination of the matrix columns. That was in fact how we defined matrix times vector in week 2, although we usually use the dot product way of computing the product entry by entry, instead:

<u>Definition</u> (from 1.4) If A is an $m \times n$ matrix, with columns $\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n$ (in $\mathbb{R}^m$) and if $\underline{\boldsymbol{x}} \in \mathbb{R}^n$, then $A\,\underline{\boldsymbol{x}}$ is defined to be the linear combination of the columns, with weights given by the corresponding entries of $\underline{\boldsymbol{x}}$. In other words,

$$A\,\underline{\boldsymbol{x}} = \left[\underline{\boldsymbol{a}}_1 \;\; \underline{\boldsymbol{a}}_2 \;\; \ldots \underline{\boldsymbol{a}}_n\right]\underline{\boldsymbol{x}} := x_1\,\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \ldots x_n\,\underline{\boldsymbol{a}}_n\,.$$

<u>Theorem</u> If we multiply a *row vector* times an $n \times m$ matrix B we get a linear combination of the *rows* of B instead. We could check this from scratch, but it's convenient to make use of transposes, which covert column facts into row facts.

$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\underline{x}^T = [x_1, x_2 \cdots x_n]$

<u>proof:</u> We want to check whether

$$* \qquad \underline{\boldsymbol{x}}^T B = \left[\, x_1 \;\; x_2 \;\; \ldots \;\; x_n \,\right] \begin{bmatrix} \underline{\boldsymbol{b}}_1 \\ \underline{\boldsymbol{b}}_2 \\ \vdots \\ \underline{\boldsymbol{b}}_n \end{bmatrix} = x_1\,\underline{\boldsymbol{b}}_1 + x_2\,\underline{\boldsymbol{b}}_2 + \ldots x_n\,\underline{\boldsymbol{b}}_n\,.$$

where the rows of B are given by the row vectors $\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n$. This proposed identity is true if and only if its transpose is a true identity. But the transpose of the left side is

$*^T$

$$\left(\underline{\boldsymbol{x}}^T B\right)^T = B^T \left(\underline{x}^T\right)^T = B^T\,\underline{\boldsymbol{x}}$$

$$= \left[\underline{\boldsymbol{b}}_1^T \;\; \underline{\boldsymbol{b}}_2^T \;\; \ldots \underline{\boldsymbol{b}}_n^T\right]\underline{\boldsymbol{x}}$$

$$x_1\,\underline{\boldsymbol{b}}_1^T + x_2\,\underline{\boldsymbol{b}}_2^T + \ldots x_n\,\underline{\boldsymbol{b}}_n^T$$

which is the transpose of the right side of the proposed identity. So the identity is true.

Q.E.D.

Exercise 2a Use the Theorem on the previous page and work row by row on so-called "elementary matrix" E_1 on the right of the product below, to show that $E_1 A$ is the result of replacing $row_3(A)$ with $row_3(A) - 2\,row_1(A)$, and leaving the other rows unchanged:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ -2a_{11}+a_{31} & -2a_{12}+a_{32} & -2a_{13}+a_{33} \end{bmatrix} = \begin{bmatrix} 1 \cdot row_1(A) \\ 1 \cdot row_2(A) \\ -2\,row_1(A) + row_3(A) \end{bmatrix}$$

2b) The inverse of E_1 must undo the original elementary row operation, so must replace any $row_3(A)$ with $row_3(A) + 2\,row_1(A)$. So it must be true that

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}. \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Check!

2c) What 3×3 matrix E_2 can we multiply times A, in order to multiply $row_2(A)$ by 5 and leave the other rows unchanged. What is E_2^{-1} ?

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2d) What 3×3 matrix E_3 can we multiply time A, in order to swap $row_1(A)$ with $row_3(A)$? What is E_3^{-1} ?

$$E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ \text{———} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$

$$E_3^{-1} = E_3$$

Definition An *elementary matrix* E is one that is obtained by doing a single elementary row operation on the identity matrix.

Theorem Let $E_{m \times m}$ be an elementary matrix. Let $A_{m \times n}$. Then the product $E\,A$ is the result of doing the same elementary row operation to A that was used to construct E from the identity matrix.

Algorithm for finding A^{-1} re-interpreted: Suppose a sequence of elementary row operations reduces the $n \times n$ square matrix A to the identity I_n. Let the corresponding elementary matrices, in order, be given by

$$E_1, E_2, \ldots \; E_p.$$

Then we have

$$E_p\left(E_{p-1} \ldots E_2\left(E_1(A)\right) \ldots\right) = I_n$$

$$\left(E_p\, E_{p-1} \ldots E_2\, E_1\right) A = I_n\,.$$

So,

$$A^{-1} = E_p\, E_{p-1} \ldots E_2\, E_1.$$

Notice that

$$E_p\, E_{p-1} \ldots E_2\, E_1 = E_p\, E_{p-1} \ldots E_2\, E_1\, I_n$$

so we have obtained A^{-1} by starting with the identity matrix, and doing the same elementary row operations to it that we did to A, in order to reduce A to the identity. I find this explanation of our original algorithm to be somewhat convoluted, but as I said, the matrix product decomposition idea pays dividends elsewhere.

Also, notice that we have ended up "factoring" A into a product of elementary matrices:

$$A = \left(A^{-1}\right)^{-1} = \left(E_p\, E_{p-1} \ldots E_2\, E_1\right)^{-1} = E_1^{-1}\, E_2^{-1} \ldots E_{p-1}^{-1}\, E_p^{-1}\,.$$

Overview of skipped section, 2.5, matrix product decompositions:

Even if A is not invertible, and even if it's not a square matrix, the elementary matrix process gives a way to factor A into a product of an invertible matrix times one that is easier to work with than A, such as a row echelon form or the reduced row echelon form. This can be helpful if one needs an alorithm to quickly and repeatedly solve $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ for a large number of right hand sides $\underline{\boldsymbol{b}}$, and there is no inverse matrix A^{-1}.

Here's how the method goes works. Let's write G for the good matrix that is, for example, the row echelon or reduced row echelon form of the $m \times n$ matrix A. Reduce as we did before for invertible matrices, to get

$$E_p\, E_{p-1} \ldots E_2\, E_1\, A = G.$$

Write

$$B = E_p\, E_{p-1} \ldots E_2\, E_1$$

for the product of elementary matrices, and note that it's the matrix one obtains by augmenting A with the $m \times m$ identity matrix and doing the same elementary row operations to it as one does to A, just as in the algorithm for constructing inverses to invertible matrices. Then

$$BA = G$$

so

$$A = B^{-1}\, G.$$

Then to solve $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ repeatedly, we want

$$B^{-1}\, G\, \underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$$

which is equivalent to the system

$$G\,\underline{\boldsymbol{x}} = B\,\underline{\boldsymbol{b}}\,.$$

Since G is a good matrix - like reduced row echelon form, this system is much quicker to solve repeatedly. These ideas are related to the concept of "preconditioning a matrix" before solving linear systems, which you can read about at Wikipedia.

Tues Feb 13

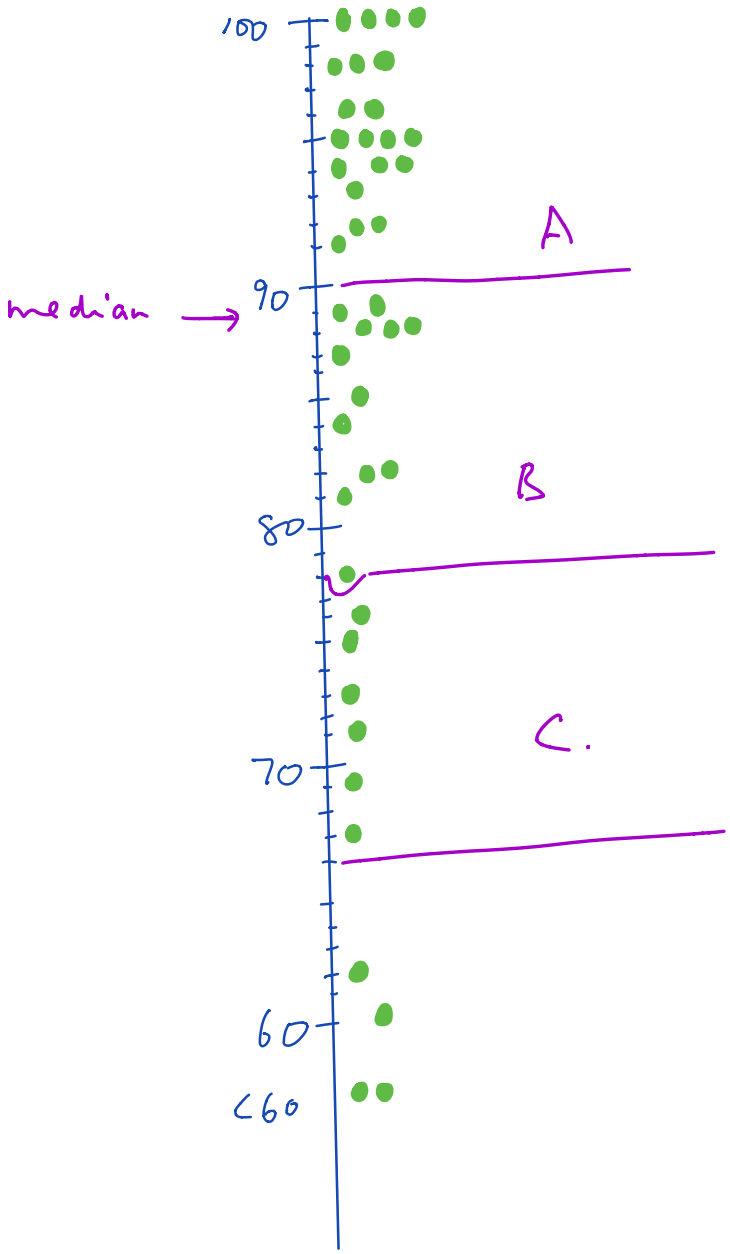
- 3.1 determinants

Announcements: look over your exams. Class did great! a few struggled.

quiz tomorrow on what we cover today & tomorrow

Warm-up Exercise: new topic – no warmup. ☺

Exam 1
2270-004
n=42



Determinants are scalars defined for square matrices $A_{n \times n}$ They always <u>determine</u> whether or not the inverse matrix A^{-1} exists, (i.e. whether the reduced row echelon form of A is the identity matrix): In fact, the determinant of A is non-zero if and only if A^{-1} exists. The determinant of a 1×1 matrix $[a_{11}]$ is defined to be the number a_{11}. (And whether or not $a_{11} = 0$ determines if it doesn't or does have a multiplicative inverese.) Determinants of 2×2 matrices are defined as in or magic formula for inverse matrices, in the 2×2 case; and in general determinants for $n \times n$ matrices are defined recursively, in terms of determinants of $(n-1) \times (n-1)$ submatrices:

<u>Definition:</u> Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

$$det(A) := \sum_{j=1}^{n} \underline{a_{1j}} \underbrace{(-1)^{1+j}}_{(\pm)} \underline{M_{1j}} = \sum_{j=1}^{n} a_{1j} C_{1j} .$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ & & & \end{bmatrix}$$

M_{11} is det of submatrix

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the jth column, and $\boxed{C_{1j} \text{ is simply } (-1)^{1+j} M_{1j}.}$

More generally, the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A is called the <u>ij Minor</u> M_{ij} of A, and $C_{ij} := (-1)^{i+j} M_{ij}$ is called the <u>ij Cofactor</u> of A.

<u>Exercise 1</u> Check that the messy looking definition above gives the same answer we talked about in regards to our formula for the inverse of 2×2 matrices, namely

"absolute value - like vertical lines, are saying determinant

know •
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} .$$

• use inductive def: $\sum_{j=1}^{2} a_{1j}(-1)^{1+j} M_{1j}$

$= a_{11} \cdot 1 \cdot a_{22}$

$+ a_{12}(-1) a_{21}$

$= a_{11}a_{22} - a_{12}a_{21}$ ✓

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

from the last page, for our convenience:

<u>Definition:</u> Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

$$\boxed{det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j} M_{1j} = \sum_{j=1}^{n} a_{1j} C_{1j}.}$$

$n=3.$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the jth column, and C_{1j} is simply $(-1)^{1+j} M_{1j}$.

<u>Exercise 2</u> Work out the expanded formula for the determinant of a 3×3 matrix. It's not worth memorizing (as opposed to the recursive formula above), but it's good practice to write out at least once, and we might point to it later.

$$\left[(-1)^{i+j}\right] = \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \end{bmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

6 terms
$6 = 3!$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

<u>note</u>: each triple product use each row & column exactly once
if order terms so that rows go 1, 2, 3
then $\pm$ is the sign of the permutation of 1, 2, 3
that gives the columns

see wikipedia

Exercise 3a) Let $A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$. Compute $det(A)$ using the definition. (On the next page we'll use other rows and columns to do the computation.)

$$|A| = 1 \cdot \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} + 2\left(-\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}\right) + (-1)\begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix}$$

$$= 1 \cdot (3 - -2) + 2(-1)(0-2) + (-1)(0-6)$$

$$= 5 + 4 + 6 = 15 \quad !$$

Theorem: $det(A)$ can be computed by expanding across any row, say row i:

$$det(A) := \sum_{j=1}^{n} a_{ij}(-1)^{i+j} M_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} \qquad row_i(A)$$

or by expanding down any column, say column j:

$$det(A) := \sum_{i=1}^{n} a_{ij}(-1)^{i+j} M_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij}. \qquad col_j(A)$$

"factor" "cofactor"

(proof is not so easy - our text skips it and so will we. If you look on Wikipedia you'll see that the determinant is actually a sum of n factorial terms, each of which is $\pm$ a product of n entries of A where each product has exactly one entry from each row and column. The $\pm$ sign has to do with whether the corresponding permutation is even or odd. You can verify this pretty easily for the 2×2 and 3×3 cases. Then one shows inductively that each row or column cofactor expansion reproduces this sum, in the $n \times n$ case.)

From previous page,

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} = \left[(-1)^{i+j}\right]$$

<u>3b)</u> Verify that the matrix of all the cofactors of A is given by $[C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$. Then expand $det(A)$ down various columns and rows using the a_{ij} factors and C_{ij} cofactors. Verify that you always get the same value for $det(A)$, as the Theorem on the previous page guarantees. Notice that in each case you are taking the dot product of a row (or column) of A with the corresponding row (or column) of the cofactor matrix.

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad [C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

did $row_1(A) \cdot row_1(cof(A))$

$row_2(A) \cdot row_2(cof(A)) = 0 + 3 \cdot 3 + 1 \cdot 6 = 15.$

$col_3(A) \cdot col_3(cof(A)) = 6 + 6 + 3 = 15$

$col_1(A) \cdot col_1(cof(A)) = 5 + 0 + 10 = 15$

$$[C_{ij}] = \begin{bmatrix} +\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} \\ +\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix} \checkmark$$

3c) What happens if you take dot products between a row of A and a *different* row of $[C_{ij}]$? A column of A and a *different* column of $[C_{ij}]$? The answer may seem magic.

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad [C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$\text{row}_1(A) \cdot \text{row}_2(\text{cof}(A)) = 0 + 6 - 6 = 0$

$\text{row}_2(A) \cdot \text{row}_3(\text{cof}(A)) = 0 - 3 + 3 = 0$

$\text{col}_2(A) \cdot \text{col}_3(\text{cof}(A)) = -12 + 18 - 6 = 0$

we always get 0!

3d) The adjoint matrix is defined to be the transpose of the cofactor matrix. So in our example,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad adj(A) = (cof(A))^T = \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}.$$

$$cof(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

Reinterpret your work in 3bc to say that

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix}.$$

So $A^{-1} = \frac{1}{|A|} adj(A)$!

So, in this case - and in fact always, the magic formula for A^{-1} is given by

$$A^{-1} = \frac{1}{det(A)} adj(A).$$

It seems like magic now, but we'll be able to understand why it's true after we learn about more determinant properties on Wednesday and Friday.

Exercise 4) Compute the following determinants by being clever about which rows or columns to use:

4a) $\begin{vmatrix} 1 & 38 & 106 & 3 \\ 0 & 2 & 92 & -72 \\ 0 & 0 & 3 & 45 \\ 0 & 0 & 0 & -2 \end{vmatrix}$; $= |A|$

4b) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ \pi^2 & 2 & 0 & 0 \\ 0.476 & 88 & 3 & 0 \\ 1 & 22 & 33 & -2 \end{vmatrix}$. $= |B|$.

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

4a) 4th row or 1st column.

$$|A| = 1 \begin{vmatrix} 2 & 92 & -72 \\ 0 & 3 & 45 \\ 0 & 0 & -2 \end{vmatrix} - 0 + 0 - 0$$

$$= 1 \cdot 2 \begin{vmatrix} 3 & 45 \\ 0 & -2 \end{vmatrix} - 0 + 0$$

$$= \underline{1 \cdot 2 \cdot 3 \cdot (-2)}$$

the matrix 

Exercise 5) Explain why it is always true that for an upper triangular matrix (as in 2a), or for a lower triangular matrix (as in 2b), the determinant is always just the product of the diagonal entries.

Wed Feb 14

- 3.2 properties of determinants

<u>Announcements:</u>

- quiz today
- fft Friday: likely about geometry of determinants

<u>Warm-up Exercise</u>: compute this determinant:

'til 10:57

$$\begin{vmatrix} 1 & 0 & 1 & 2 \\ 2 & 0 & -4 & 2 \\ 7 & 3 & 8 & 6 \\ 0 & 0 & -3 & 0 \end{vmatrix}$$

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

col_2 or row_4 are expansion choices.

$|A|$

across any row i:

$$\sum_{j=1}^{n} a_{ij} \underbrace{C_{ij}}_{(-1)^{i+j} M_{ij}}$$

or down column j:

$$\sum_{i=1}^{n} a_{ij} C_{ij}$$

col_2: $(-0)\cdot M_{12}$

$+0\, M_{22}$

$-3\, M_{32} + 0\, M_{42}$.

$$= -3 \begin{vmatrix} 1 & 1 & 2 \\ 2 & -4 & 2 \\ 0 & -3 & 0 \end{vmatrix}$$

$$= -3\left(0 - (-3)\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} + 0 \cdot \right)$$

$$= -9(2-4) = 18$$

The effective way to compute determinants for larger-sized matrices without lots of zeroes is to not use the definition, but rather to use the following facts, which track how elementary row operations affect determinants:

- (1a) Swapping any two rows changes the sign of the determinant.
  proof: This is clear for 2×2 matrices, since

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad .$$

  For 3×3 determinants, expand across the row *not* being swapped, and use the 2×2 swap property to deduce the result. Prove the general result by induction: once it's true for $n \times n$ matrices you can prove it for any $(n+1) \times (n+1)$ matrix, by expanding across a row that wasn't swapped, and applying the $n \times n$ result.

eg. swap row_1 & row_2

unswapped:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = +a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

what happens on the right if I swap row1 & row2 on the left?

Ans: rows swap on RHS, so get opposite values from before.

so entire 3×3 determinant changes sign.

so by induction, can show if statement is true for all $n\times n$ matrices, it's also true for $(n+1)\times(n+1)$ matrices.

(1b) Thus, if two rows in a matrix are the same, the determinant of the matrix must be zero: on the one hand, swapping those two rows leaves the matrix and its determinant unchanged; on the other hand, by (1a) the determinant changes its sign. The only way this is possible is if the determinant is zero.

- (2a) If you factor a constant out of a row, then you factor the same constant out of the determinant. Precisely, using $\mathcal{R}_i$ for i^{th} row of A, and writing $\mathcal{R}_i = c\,\mathcal{R}_i^*$

$$\begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_i \\ \mathcal{R}_n \end{vmatrix} = \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ c\,\mathcal{R}_i^* \\ \mathcal{R}_n \end{vmatrix} = c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_i^* \\ \mathcal{R}_n \end{vmatrix} .$$

i^{th} row $\longrightarrow$

$[a_{i1}\ a_{i2}\ \cdots\ a_{in}]$

$= [c a_{i1}^*\ c a_{i2}^*\ \cdots\ c a_{in}^*]$

proof: expand across the ith row, noting that the corresponding cofactors don't change, since they're computed by deleting the ith row to get the corresponding minors:

$$det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} c\, a_{ij}^* C_{ij} = c \sum_{j=1}^{n} a_{ij}^* C_{ij} = c\, det(A^*) .$$

expand across i^{th} row.

$$\begin{vmatrix} 2 & 4 \\ 9 & 3 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 \\ 9 & 3 \end{vmatrix} = 2 \cdot 3 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 2 \cdot 3\,(1-6) = 2 \cdot 3(-5) = -30$$

$\|$

$6 - 36 = -30$

(2b) Combining (2a) with (1b), we see that if one row in A is a scalar multiple of another, then $det(A) = 0$.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 8 \\ 1 & 0 & 2 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{vmatrix} = 0 \text{ by } 1b$$

- (3) If you replace row i of A, $\mathcal{R}_i$ by its sum with a multiple of another row, say $\mathcal{R}_k$ then the determinant is unchanged! Expand across the ith row:

$$\begin{array}{r} \\ \\ \\ i^{th} \text{ row location} \\ \\ \end{array} \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_i + c\,\mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} = \sum_{j=1}^{n} \underbrace{(a_{ij} + c\,a_{kj})}_{\substack{\text{entries} \\ \text{in } i^{th} \\ \text{row of} \\ \text{matrix} \\ \text{on left}}} C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} + c \sum_{j=1}^{n} a_{kj} C_{ij} = det(A) + c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} = det(A) + 0 \,.$$

<u>Remark:</u> The analogous properties hold for corresponding "elementary column operations". In fact, the proofs are almost identical, except you use column expansions.

Exercise 1) Recompute $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$ from yesterday (using row and column expansions we always got an answer of 15 then.) This time use elementary row operations (and/or elementary column operations).

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$$

$$= \underset{-2R_1+R_3 \to R_3}{} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & -6 & 3 \end{vmatrix} = \underset{2R_2+R_3 \to R_3}{} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{vmatrix} = 1 \cdot 3 \cdot 5$$

upper Δ'ular matrix

$$5 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 15$$

$$\parallel$$

$$\cancel{= \text{factor } 3 \text{ out of row } 3} \quad \cancel{= 3 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & -2 & 1 \end{vmatrix}}$$

expand down 1st column.

$$3 \left(1 \cdot \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} - 0 + 0 \right)$$

$$= 3(3+2) = 15 \checkmark$$

Exercise 2) Compute $\begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ -1 & 0 & -2 & 1 \end{vmatrix}$.

Theorem: Let $A_{n \times n}$. Then A^{-1} exists if and only if $det(A) \neq 0$.

proof: We already know that A^{-1} exists if and only if the reduced row echelon form of A is the identity matrix. Now, consider reducing A to its reduced row echelon form, and keep track of how the determinants of the corresponding matrices change: As we do elementary row operations,

- if we swap rows, the sign of the determinant switches.
- if we factor non-zero factors out of rows, we factor the same factors out of the determinants.
- if we replace a row by its sum with a multiple of another row, the determinant is unchanged.

Thus,

$$|A| = c_1|A_1| = c_1c_2|A_2| = \ldots = c_1c_2 \ldots c_N|rref(A)|$$

where the nonzero c_k's arise from the three types of elementary row operations. If $rref(A) = I$ its determinant is 1, and $|A| = c_1c_2 \ldots c_N \neq 0$. If $rref(A) \neq I$ then its bottom row is all zeroes and its determinant is zero, so $|A| = c_1c_2 \ldots c_N(0) = 0$. Thus $|A| \neq 0$ if and only if $rref(A) = I$ if and only if A^{-1} exists !

Theorem: Using the same ideas as above, we can show that $det(AB) = det(A)det(B)$. This is an important identity that gets used, for example, in multivariable change of variables formulas for integration, using the Jacobian matrix. (It is not true that $det(A + B) = det(A) + det(B)$.)

Here's how to show $det(AB) = det(A)det(B)$: The key point is that if you do an elementary row operation to AB, that's the same as doing the elementary row operation to A, and then multiplying by B. With that in mind, if you do exactly the same elementary row operations as you did for A in the theorem above, you get

$$|AB| = c_1|A_1B| = c_1c_2|A_2B| = \ldots = c_1c_2 \ldots c_N|rref(A)B|.$$

If $rref(A) = I$, then from the theorem above, $|A| = c_1c_2 \ldots c_N$, and we deduce $|AB| = |A||B|$. If $rref(A) \neq I$, then its bottom row is zeroes, and so is the bottom row of $rref(A)B$. Thus $|AB| = 0$ and also $|A||B| = 0$.

Fri Feb 14

- 3.3 adjoint formula for inverses, Cramer's rule, geometric meanings of determinants.

Announcements: for last 25 minutes fft geometric meaning of determinants due Tues.

Warm-up Exercise: 'til 12:57

Compute the following two determinants by expanding across the 2nd row.

Notice that

1) The cofactors are the same. because you crossed out row 2

2) You know that one of the determinants has to be zero

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix} = -0\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = 0 + 3\cdot 3 - 1(-6) = \underline{\underline{15}}$$

$$\begin{vmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 2 & -2 & 1 \end{vmatrix} = 0 = -1\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix}$$

2 rows are the same

$$= -1(0) + 2(3) + 1(-6) = 0!$$

Theorem: Let $A_{n \times n}$, and denote its cofactor matrix by $cof(A) = [C_{ij}]$, with $C_{ij} = (-1)^{i+j} M_{ij}$, and $M_{ij} =$ the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A.

formal

Define the adjoint matrix to be the transpose of the cofactor matrix:

$$Adj(A) := cof(A)^T$$

Then, when A^{-1} exists it is given by the formula

$$A^{-1} = \frac{1}{det(A)} Adj(A) .$$

Exercise 1) Show that in the 2×2 case this reproduces the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} .$$

$$\begin{matrix} + & - \\ - & + \end{matrix}$$

$$cof(A) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$Adj(A) = cof(A)^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \checkmark$$

Example) For our friend $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$ we worked out $cof(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$ so

transpose.

$5 = +\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix}$,

$2 = -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}$

etc.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \quad adj(A) = \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix} \quad det(A) = 15,$$

so

$A \cdot Adj(A) = \begin{bmatrix} 15 \\ \\ \end{bmatrix}$

$$A^{-1} = \frac{1}{15}\begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}$$

Let's understand why the magic worked:

Exercise 2) Continuing with our example,

2a) The $(1, 1)$ entry of $(A)(Adj(A))$ is $15 = 1 \cdot 5 + 2 \cdot 2 + (-1)(-6)$. Explain why this is $det(A)$, expanded across the first row.

2b) The $(2, 1)$ entry of $(A)(Adj(A))$ is $0 \cdot 5 + 3 \cdot 2 + (1)(-6) = 0$. Notice that you're using the same cofactors as in (2a). What matrix, which is obtained from A by keeping two of the rows, but replacing a third one with one of those two, is this the determinant of?

$row_2(A) \cdot row_1(cof(A)) = \begin{vmatrix} 0 & 3 & 1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$ $\leftarrow row_2(A)$ } unchanged.

2c) The $(3, 2)$ entry of $(A)(Adj(A))$ is $2 \cdot 0 - 2 \cdot 3 + 1 \cdot 6 = 0$. What matrix (which uses two rows of A) is this the determinant of?

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{vmatrix} = 0$$

If you completely understand 2abc, then you have realized why

$$[A][Adj(A)] = det(A)[I]$$

for every square matrix, and so also why

$$A^{-1} = \frac{1}{det(A)} Adj(A) .$$

Precisely,

$$entry_{i\,i}\, A(Adj(A)) = row_i(A) \bullet col_i(Adj(A)) = row_i(A) \bullet row_i(cof(A)) = det(A),$$

expanded across the ith row.

On the other hand, for $i \neq k$,

$$entry_{k\,i}\, A(Adj(A)) = row_k(A) \bullet col_i(Adj(A)) = row_k(A) \bullet row_i(cof(A)) .$$

This last dot produce is zero because it is the determinant of a matrix made from A by replacing the ith row with the kth row, expanding across the ith row, and whenever two rows are equal, the determinant of a matrix is zero:

$$\begin{array}{r} \\ \\ i^{th} \text{ row position} \\ \\ \\ \end{array} \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} .$$

There's a related formula for solving for individual components of $\underline{\boldsymbol{x}}$ when $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ has a unique solution ($\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$). This can be useful if you only need one or two components of the solution vector, rather than all of it:

<u>Cramer's Rule</u>: Let $\underline{\boldsymbol{x}}$ solve $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$, for invertible A. Then

$$x_k = \frac{det(A_k)}{det(A)}$$

where A_k is the matrix obtained from A by replacing the kth column with $\underline{\boldsymbol{b}}$.

proof: Since $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$ the kth component is given by

$$\begin{aligned} x_k &= entry_k\left(A^{-1}\underline{\boldsymbol{b}}\right) \\ &= entry_k\left(\frac{1}{|A|}Adj(A)\underline{\boldsymbol{b}}\right) \\ &= \frac{1}{|A|}row_k(Adj(A)) \cdot \underline{\boldsymbol{b}} \\ &= \frac{1}{|A|}col_k(cof(A)) \cdot \underline{\boldsymbol{b}} . \end{aligned}$$

Notice that $col_k(cof(A)) \cdot \underline{\boldsymbol{b}}$ is the determinant of the matrix obtained from A by replacing the kth column by $\underline{\boldsymbol{b}}$, where we've computed that determinant by expanding down the kth column! This proves the result. (See our text for another way of justifying Cramer's rule.)

<u>Exercise 3)</u> Solve $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

<u>3a)</u> With Cramer's rule

<u>3b)</u> With A^{-1}, using the adjoint formula. (skip).

$$x = \frac{\begin{vmatrix} 7 & -1 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 5 & -1 \\ 4 & 1 \end{vmatrix}} = \frac{9}{9} = 1$$

$$y = \frac{\begin{vmatrix} 5 & 7 \\ 4 & 2 \end{vmatrix}}{\begin{vmatrix} 5 & -1 \\ 4 & 1 \end{vmatrix}} = \frac{-18}{9} = -2.$$

$$\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix} \checkmark$$

Friday Food for Thought 5

Due Tuesday February 20

Spend the rest of today's class period working through these problems. I encourage you to work with your classmates and discuss the problems. If you are finished with the assignment at the end of class today, then you can turn it in today. If you would like to work on the assignment more, take it home over the weekend and turn it in on Tuesday. This assignment will be graded for ***effort*** *(which means that you have written down thoughtful, complete solutions to each problem), not correctness. Solutions to these problems will be posted on Canvas Tuesday for future reference.*

Let's explore what determinants have to do with linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ (generalizes to the case of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$), and with *affine transformations*, which are compositions of translations and linear transformations. So for today, we'll be thinking about functions of the form

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

which transform $\mathbb{R}^2$ to $\mathbb{R}^2$. Since

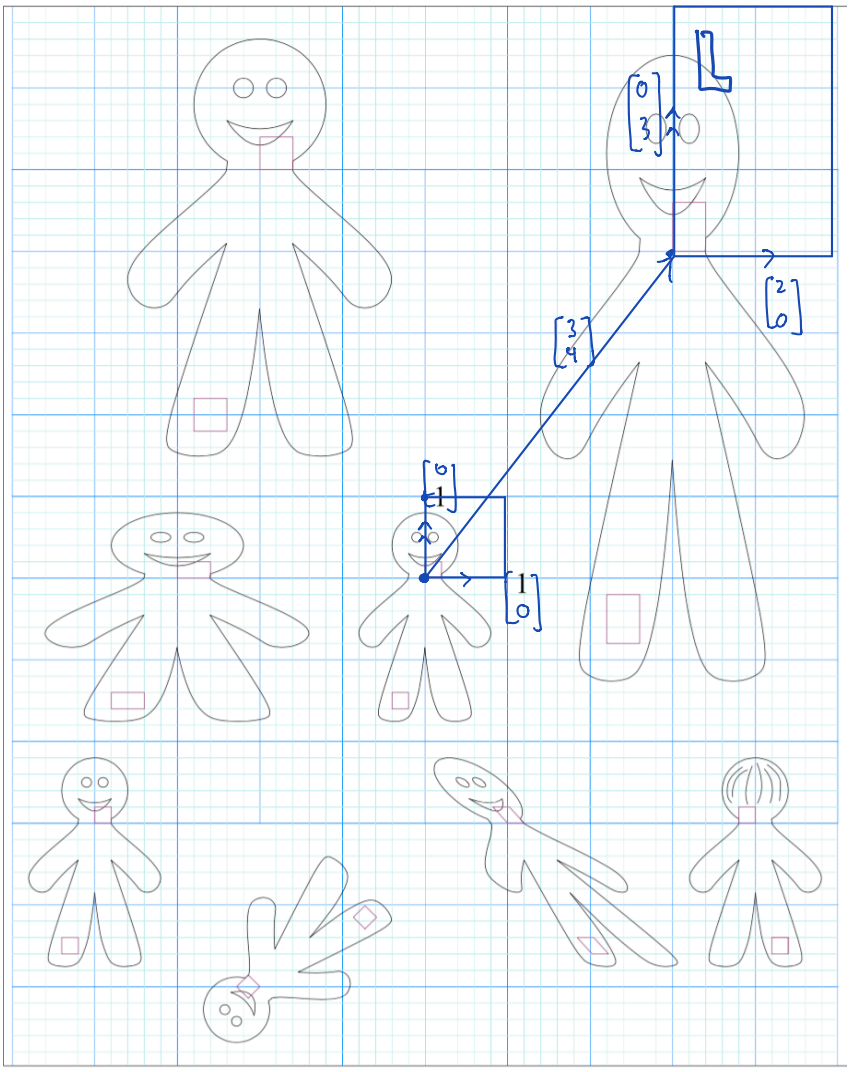
$$F\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$F\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

$$F\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix},$$

You can reconstruct the formula for the affine function as soon as you know the images of $\underline{\mathbf{0}}, \underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2$. For example, I reconstructed the transformation formula for Giant Bob in the upper right corner of the next page. Notice that Giant Bob has six times the area of original Bob - since original Bob can be filled up with different-sized squares, and the images of those squares will be rectangles having six times the original areas.

1) Reconstruct the formulas for at least three more of the six (non-identity) transformations of Bob on the next page, and comment on how the areas of the transformed Bobs are related to the determinants of the matrices in the transformations. Note that the Bob in the lower right corner got flipped over.



$$G\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}3\\4\end{bmatrix}$$

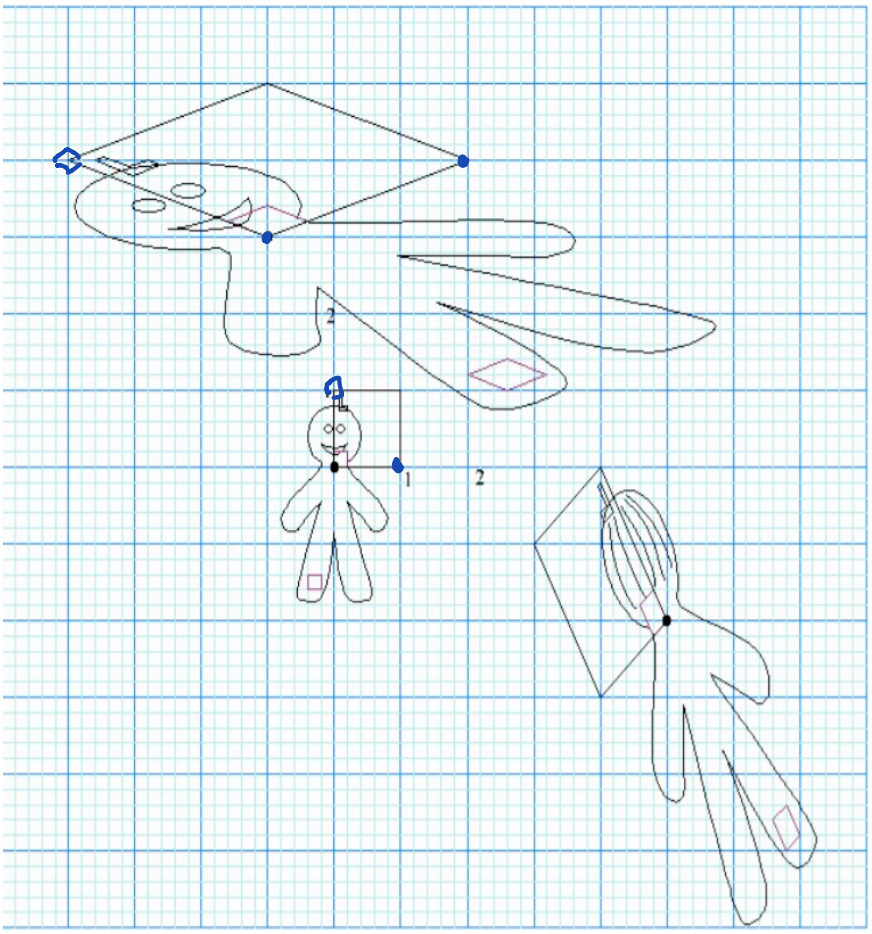
$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

$$F\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} e \\ f \end{bmatrix} \quad 1^{st}$$

$$F\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} \quad \text{for } \begin{bmatrix} a \\ b \end{bmatrix}$$

$$F\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} \quad \text{for } \begin{bmatrix} c \\ d \end{bmatrix}$$

2) Find the formulas for these two affine transformations of Bob.



3) Squares in original Bob get transformed into parallelegrams in the image Bobs, and the area expansion factors are independent of the size of the original squares. So, you can deduce the area expansion factor for the image Bobs just by computing the area of the parallegram image of the unit square. How do your area expansion factors in these two examples compare to the matrix determinants from the affine transformations?

We'll talk more systematically about area/volume expansion factors, and in arbitrary dimension on Tuesday, but for affine transformations from $\mathbb{R}^2 \to \mathbb{R}^2$ one can use geometry to connect determinants to area expansion factors.

4) Can you compute the area of the parallelgram below (in terms of the letters a, b, c, d)? Since translations don't effect area, this will give the area expansion factor also for the images of arbitrary regions, under affine transformations

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}.$$

Hint: Start with the area of the large rectangle of length $a + c$ and height $b + d$, then subtract off the areas of the triangles and rectangles on the outside of the parallelgram. For convenience I chose the case where all of a, b, c, d are positive, and where the transformation didn't "flip" the parallelgram: