Math 2270-004 Week 5 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 2.1-2.3.

Mon Feb 5
- 2.1-2.2 Matrix algebra and matrix inverses

Announcements: finish algebra from Friday's notes
start inverse matrices.
$\left.\begin{matrix}2.1\\2.2\end{matrix}\right\}$ today

Warm-up Exercise: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ 3/2 & -\frac{1}{2} \end{bmatrix}$

'til 12:58

compute AB and BA

$AB = BA = I_{2\times 2} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I\vec{x} = \vec{x}$
identity matrix I.

$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} -2+3 & 1-1 \\ -6+6 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

we would write $B = A^{-1}$, because $AB = BA = I$

for functions, if $T_1(\vec{x}) = A\vec{x}$

$T_2(\vec{y}) = B\vec{y}$

$T_2 \circ T_1(\vec{x}) = B(A\vec{x}) = (BA)\vec{x} = I\vec{x} = \vec{x}$

also $T_1 \circ T_2(\vec{y}) = \vec{y}$.

So the transformation function T_1 has inverse transformation T_2

Remember from last week that matrices correspond to linear transformations, and that products of matrices are the matrices of compositions of linear transformations:

2.1 Matrix multiplication

Suppose we take a composition of linear transformations:

$$T_1 : \mathbb{R}^n \to \mathbb{R}^m, \quad T_1(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}, \qquad (A_{m \times n}).$$

$$T_2 : \mathbb{R}^m \to \mathbb{R}^p, \quad T_2(\underline{\boldsymbol{y}}) = B\,\underline{\boldsymbol{y}}. \qquad (B_{p \times m})$$

- Then the composition $T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^p$ is linear:

(i)
$$\begin{aligned}
(T_2 \circ T_1)(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) &:= \left(T_2\left(T_1(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}})\right)\right) \\
&= T_2\left(T_1(\underline{\boldsymbol{u}}) + T_1(\underline{\boldsymbol{v}})\right) && T_1 \text{ linear} \\
&= T_2\left(T_1(\underline{\boldsymbol{u}})\right) + T_2\left(T_1(\underline{\boldsymbol{v}})\right) && T_2 \text{ linear} \\
&:= (T_2 \circ T_1)(\underline{\boldsymbol{u}}) + (T_2 \circ T_1)(\underline{\boldsymbol{v}})
\end{aligned}$$

(ii)
$$\begin{aligned}
(T_2 \circ T_1)(c\,\underline{\boldsymbol{u}}) &:= \left(T_2\left(T_1(c\,\underline{\boldsymbol{u}})\right)\right) \\
&= T_2\left(c\,T_1(\underline{\boldsymbol{u}})\right) && T_1 \text{ linear} \\
&= c\,T_2\left(T_1(\underline{\boldsymbol{u}})\right) && T_2 \text{ linear} \\
&:= c\,(T_2 \circ T_1)(\underline{\boldsymbol{u}})
\end{aligned}$$

- So $T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^p$ is a matrix transformation, by the theorem on the previous page.

- Its j^{th} column is

$$\begin{aligned}
T_2 \circ T_1(\vec{e}_j) = T_2\left(T_1(\underline{\boldsymbol{e}}_j)\right) &= T_2\left(A(\underline{\boldsymbol{e}}_j)\right) \\
&= T_2\left(col_j(A)\right) \\
&= B\left(col_j(A)\right).
\end{aligned}$$

- i.e. the matrix of $T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^p$ is

$$\left[B\,\underline{\boldsymbol{a}}_1 \; B\,\underline{\boldsymbol{a}}_2 \; \ldots \; B\,\underline{\boldsymbol{a}}_n\right] := B\,A.$$

where $col_j(A) = \underline{\boldsymbol{a}}_j$.

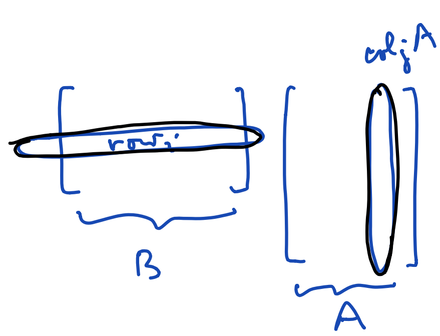
$\boxed{\text{Summary:}}$ For $B_{p \times m}$ and $A_{m \times n}$

- the matrix product $(BA)_{p \times n}$ is defined by

$$col_j(B\,A) = B\,col_j(A) \qquad j = 1 \mathinner{..} n$$

- or equivalently

$$entry_{ij}(B\,A) = row_i(B) \cdot col_j(A) \quad i = 1 \mathinner{..} p, j = 1 \mathinner{..} n$$



- And,

$$(BA)\underline{\boldsymbol{x}} = B\,(A\,\underline{\boldsymbol{x}})$$

because BA is the matrix of $T_2 \circ T_1$.

$entry_{ij}(BA)$
$= row_i(B) \cdot col_j(A)$

2.1 matrix algebra....we already talked about matrix multiplication. It interacts with matrix addition in interesting ways. We can also add and scalar multiply matrices of the same size, just treating them as oddly-shaped vectors:

<u>Matrix algebra:</u>

- <u>addition and scalar multiplication:</u> Let $A_{m \times n}$, $B_{m \times n}$ be two matrices of the same dimensions (m rows and n columns). Let $entry_{i\,j}(A) = a_{i\,j}$, $entry_{i\,j}(B) = b_{i\,j}$. (In this case we write $A = \left[a_{i\,j}\right]$, $B = \left[b_{i\,j}\right]$.) Let c be a scalar. Then

$$entry_{i\,j}(A + B) := a_{i\,j} + b_{i\,j}.$$
$$entry_{i\,j}(c\,A) := c\,a_{i\,j}.$$

In other words, addition and scalar multiplication are defined analogously as for vectors. In fact, for these two operations you can just think of matrices as vectors written in a rectangular rather than row or column format.

<u>Exercise 1)</u> Let $A := \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 0 & 3 \end{bmatrix}$ and $B := \begin{bmatrix} 0 & 27 \\ 5 & -1 \\ -1 & 1 \end{bmatrix}$. Compute $4\,A - B$.

$$4A - B = 4\begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 27 \\ 5 & -1 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -8 \\ 12 & -4 \\ 0 & 12 \end{bmatrix} - \begin{bmatrix} 0 & 27 \\ 5 & -1 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -35 \\ 7 & -3 \\ 1 & 11 \end{bmatrix}$$

<u>vector properties of matrix addition and scalar multiplication</u>

But other properties you're used to do hold:

- $+$ is commutative $\qquad A + B = B + A$

$$entry_{ij}(A + B) = a_{ij} + b_{ij} = b_{ij} + a_{ij} = entry_{ij}(B + A)$$

- $+$ is associative $\qquad (A + B) + C = A + (B + C)$

the ij entry of each side is $a_{ij} + b_{ij} + c_{ij}$

- scalar multiplication distributes over $+$ $\qquad c(A + B) = cA + cB$.

ij entry of LHS is $c\left(a_{ij} + b_{ij}\right) = c\left(a_{ij} + b_{ij}\right) = ij$ *entry of RHS*

More interesting are how matrix multiplication and addition interact:

Check some of the following. Let I_n be the $n \times n$ identity matrix, with $I_n \underline{x} = \underline{x}$ for all $\underline{x} \in \mathbb{R}^n$. Let A, B, C have compatible dimensions so that the indicated expressions make sense. Then

a $A(BC) = (AB)C$ (associative property of multiplication)

$A_{m\times n}(B_{n\times p}C_{pq})$ $\quad (A_{m\times n}B_{n\times p})C_{pq}$

$\underbrace{A_{m\times n}(BC)_{n\times q}}_{[A(BC)]_{m\times q}}$ $\quad (AB)_{m\times p}C_{p\times q}$ $\quad [(AB)C]_{m\times q}$

look at j^{th} columns:

$$\begin{aligned} col_j(A(BC)) &= A\, col_j(BC) \\ &= A(B\, col_j C) \\ &= (AB)\, col_j C \\ &= col_j[(AB)C] \end{aligned}$$

because of composition rule!!

b $A(B + C) = AB + AC$ (left distributive law)

Monday check.

compare the j^{th} columns.

$$\begin{aligned} col_j\, A(B+C) := A\, col_j(B+C) &= A(col_j(B) + col_j(C)) \\ &= A\, col_j(B) + A\, col_j(C) \\ &= col_j(AB) + col_j(AC) \\ &= col_j(AB+AC) \end{aligned}$$

matrix mult. is linear

c $(A + B)C = AC + BC$ (right distributive law)

d $rAB = (rA)B = A(rB)$ for any scalar r.

$entry_{ij}$ (each of these)

$= r\, entry_{ij}(AB)$

e.g.

$$r\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\right) = r\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}\end{bmatrix}$$

$$= \begin{bmatrix} r(\text{——}) & r(\text{——}) \\ r(\text{——}) & r(\text{——})\end{bmatrix}$$

$$(rA)B = \begin{bmatrix} ra_{11} & ra_{12} \\ ra_{21} & ra_{22}\end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix} = \text{same.}$$

$A_{m\times n} I_{n\times n}$

e) If $A_{m \times n}$ then $I_m A = A$ and $A I_n = A$.

Warning: $AB \neq BA$ in general. In fact, the sizes won't even match up if you don't use square matrices.

The transpose operation:

Definition: Let $B_{m \times n} = \left[b_{i j} \right]$. Then the transpose of B, denoted by B^T is an $n \times m$ matrix defined by

$1 \le i \le m$
$1 \le j \le n$

$$entry_{i j}\left(B^T\right) := entry_{j i}(B) = b_{j i} .$$

$1 \le j \le n$
$1 \le i \le m$

The effect of this definition is to turn the columns of B into the rows of B^T:

$$entry_i\left(col_j(B)\right) = b_{i j} .$$
$$entry_i\left(row_j\left(B^T\right)\right) = entry_{j i}\left(B^T\right) = b_{i j} .$$

j^{th} column of $B = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix}$

And to turn the rows of B into the columns of B^T:

$$entry_j\left(row_i(B)\right) = b_{i j}$$
$$entry_j\left(col_i\left(B^T\right)\right) = entry_{j i}\left(B^T\right) = b_{i j} .$$

j^{th} row of B^T
$= \left[b^T_{j1} \;\; b^T_{j2} \; -- \; b^T_{jm} \right]$
$= \left[b_{1j} \;\; b_{2j} \; -- \; b_{mj} \right]$ ✓

(so j^{th} col. of B became j^{th} row of B^T)

Exercise 2) explore these properties with the identity

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} .$$

Algebra of transpose:

a $\left(A^T\right)^T = A$ ✓

b $(A + B)^T = A^T + B^T$ ✓

c for every scalar r $(rA)^T = r A^T$ ✓

d (The only surprising property, so we should check it.) $(A B)^T = B^T A^T$

$entry_{ij}(AB)^T := entry_{ji}(AB)$
$= row_j(A) \cdot col_i(B)$

$entry_{ij}(B^T A^T) = row_i(B^T) \cdot col_j(A^T)$

same because
$row_j(A)$ has
same entries as
$col_j(A^T)$
$col_i(B)$ same entries as
$row_i(B^T)$.

2.2 matrix inverses.

We've been talking about matrix algebra: addition, scalar multiplication, multiplication, and how these operations combine. But I haven't told you what the algebra on the previous page is good for. Today we'll start to find out. By way of comparison, think of a scalar linear equation with known numbers *a*, *b*, *c*, *d* and an unknown number *x*,

$$a\,x + b = c\,x + d$$

We know how to solve it by collecting terms and doing scalar algebra:

$$a\,x - c\,x = d - b$$

$$(a - c)\,x = d - b \qquad *$$

$$x = \frac{d - b}{a - c}\,.$$

How would you solve such an equation if *A*, *B*, *C*, D were square matrices, and *X* was a vector (or matrix) ? Well, you could use the matrix algebra properties we've been discussing to get to the $*$ step. And then if *X* was a vector you could solve the system * with Gaussian elimination. In fact, if *X* was a matrix, you could solve for each column of *X* (and do it all at once) with Gaussian elimination.

But you couldn't proceed as with scalars and do the final step after the * because it is not possible to divide by a matrix. Today we'll talk about a potential shortcut for that last step that is an analog of of dividing, in order to solve for *X*. It involves the concept of *inverse matrices*.

$$AX + B = CX + D$$

$$(AX + B) - CX = (CX + D) - CX$$

$$[(AX - CX) + \cancel{B}] \cancel{-B} = [(\cancel{CX - CX}) + D] - B$$

$$\vdots$$

$$(A - C)X = D - B$$

← we could solve by reducing a very augmented matrix.

but you cannot divide by a matrix, can't do *.

but if $(A-C)$ had an inverse matrix, you could do something almost as good as division.

Matrix inverses: A square matrix $A_{n \times n}$ is invertible if there is a matrix $B_{n \times n}$ so that

$$AB = BA = I .$$

In this case we call B the inverse of A, and write $B = A^{-1}$.

Remark 1: A matrix A can have at most one inverse, because if we have two candidates B, C with

$$AB = BA = I \quad \text{and also} \quad AC = CA = I$$

then

$$(BA)C = IC = C$$
$$B(AC) = BI = B$$

so since the associative property $(BA)C = B(AC)$ is true, it must be that

$$B = C.$$

Remark 2: In terms of linear transformations, if $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the linear transformation $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$, then saying that A has an inverse matrix is the same as saying that T has an inverse linear transformation, $T^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with matrix B so that $T^{-1} \circ T(\underline{\boldsymbol{x}}) = \underline{\boldsymbol{x}} \quad \forall\, \underline{\boldsymbol{x}} \in \mathbb{R}^n$ and $T \circ T^{-1}(\underline{\boldsymbol{y}}) = \underline{\boldsymbol{y}} \;\; \forall\, \underline{\boldsymbol{y}} \in \mathbb{R}^n$. Your final food for thought question from last Friday explains why linear transformations $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ only have a chance at having inverse transforms when $n = m$.

Exercise 1a) Verify that for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ the inverse matrix is $A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$. $= B$

this was your warmup!

Tuesday:

$$AB = BA = I$$

Where did the matrix A^{-1} come from? How to find??

Inverse matrices can be useful in solving algebra problems. For example

Theorem: If A^{-1} exists then the only solution to $A\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ is $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$.

Exercise 1b) Use the theorem and A^{-1} in 2a, to write down the solution to the system

$$x + 2y = 5$$
$$3x + 4y = 6$$

$-4+9=5$ ✓
$-12+18=6$ ✓.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

ans is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -4 \\ \frac{15}{2} - \frac{6}{2} \end{bmatrix} = \begin{bmatrix} -4 \\ \frac{9}{2} \end{bmatrix}$

Exercise 2a) Use matrix algebra to verify why the Theorem above is true. Notice that the correct formula is $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$ and not $\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}\,A^{-1}$. (This second product can't even be computed because the dimensions don't match up!)

If $A\vec{x} = \vec{b}$

then $A^{-1}(A\vec{x}) = A^{-1}\vec{b}$

$(A^{-1}A)\vec{x} = A^{-1}\vec{b}$

$I\vec{x} = A^{-1}\vec{b}$

$\underline{\vec{x} = A^{-1}\vec{b}}$

also, check ans:

$A(A^{-1}\vec{b}) = (AA^{-1})\vec{b}$

$= I\vec{b}$

$= \vec{b}$ ✓

Corollary: If A^{-1} exists, then the reduced row echelon form of A is the identity matrix.
proof: For a square matrix, solutions to $A\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$ always exist and are unique only when A reduces to the identity. When A^{-1} exists, the solutions to $A\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$ exist and are unique, so $rref(A)$ must equal the identity.

know if A^{-1} exists, the soltn to $A\vec{x} = \vec{b}$
is $\vec{x} = A^{-1}\vec{b}$

solutions are unique

$\Rightarrow rref(A) = I$

$$A\vec{x} = \vec{b}$$
$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

Exercise 3 Assuming A is a square matrix with an inverse A^{-1}, and that the matrices in the equation below have dimensions which make for meaningful equation, solve for X in terms of the other matrices:

$$XA + C = B$$

$$\Leftrightarrow \quad XA = B - C$$

careful which side I multiply on!

$$\Leftrightarrow \quad (X(A)A^{-1}) = (B - C)A^{-1}$$

$$X = (B - C)A^{-1}$$

But where did that formula for A^{-1} come from?

<u>One Answer:</u> Consider A^{-1} as an unknown matrix, $A^{-1} = X$. We want

$$A\,X = I\,.$$

We can break this matrix equation down by the columns of X. In the two by two case we get:

$$A\left[\,col_1(X)\,\middle|\,col_2(X)\,\right] = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right].$$

$$X = \left[\begin{array}{c:c} x_1 & y_1 \\ x_2 & y_2 \end{array}\right].$$

In other words, the two columns of the inverse matrix X should satisfy

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\left(col_1(X)\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A\left(col_2(X)\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. = A\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

We can solve for both of these mystery columns at once, as we've done before when we had different right hand sides:

to find col_1 of A^{-1}

to find col_2.

<u>Exercise 3:</u> Reduce the double augmented matrix

$$\left[\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

to find the two columns of A^{-1} for the previous example.

$$\Big\downarrow \text{rref}$$

$$\left[\begin{array}{cc|c|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

$x_1 = -2$
$x_2 = 3/2$

$y_1 = 1$
$y_2 = -1/2$

so $X = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

$AX = I$ ✓

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

($XA = I$ as well, so
$A^{-1} = X$

For 2×2 matrices there's also a cool formula for inverse matrices:

<u>Theorem:</u> $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ exists if and only if the <u>determinant</u> $D = ad - bc$ of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-zero. And in this case,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Notice that the diagonal entries have been swapped, and minus signs have been placed in front of the off-diagonal terms. This formula should be memorized.)

<u>Exercise 4)</u> Check that the magic formula reproduces the answer you got in Exercise 3 for

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}$$

$$\frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{4-6} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

<u>Remark)</u> If $ad - bc = 0$ then rows of A are multiples of each other, so A cannot reduce to the identity, so doesn't have an inverse matrix.

<u>Exercise 4:</u> Will this always work? Can you find A^{-1} for

$$A := \begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix} ?$$

$$AX = I$$

$$A \left[\begin{array}{c:c:c} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{array}\right] = \left[\begin{array}{c:c:c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 7 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_3 + R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ \\ 3R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & -1 & 4 & 3 & -5 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right]$$

$$\begin{array}{r} -R_3 + R_1 \to R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 5 & 3 & -5 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right]$$

$$\begin{array}{r} -5R_2 + R_1 \to R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -2 & 5 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right] \quad A^{-1}$$

$x_1 = -5$, $x_2 = 2$, $x_3 = -4$

$y_1 = -2$, $y_2 = 1$, $y_3 = -3$

$z_1 = 5$, $z_2 = -2$, $z_3 = 5$

check!

$$AA^{-1} = I$$

Exercise 5) Will this always work? Try to find B^{-1} for $B := \begin{bmatrix} 1 & 5 & 5 \\ 2 & 5 & 0 \\ 2 & 7 & 4 \end{bmatrix}$.

Here's what happens when we try to find the three columns of B^{-1}:

$$BaugI := \left[\begin{array}{ccc|ccc} 1 & 5 & 5 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 7 & 4 & 0 & 0 & 1 \end{array}\right] \qquad rref(BaugI) = \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 0 & \frac{7}{4} & -\frac{5}{4} \\ 0 & 1 & 2 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{3}{4} & -\frac{5}{4} \end{array}\right]$$

$\rightsquigarrow$

rref$(B) \neq I$, B^{-1} does not exist

(and systems for ~~at~~ at least col. of B^{-1} is inconsistent.
In this, for all 3 columns

Tues Feb 6

- 2.2-2.3 Matrix inverses

Announcements:

- no quiz tomorrow
- Review session for Friday midterm is $\boxed{\text{Thursday 12:55-2:15 JWB 335}}$
- fft #1, #2 → everyone got 10/10 who handed it in
  #3, #4 I actually made suggestions
  solutions posted on CANVAS → check for correctness!
  midterm will have a mixture of computational & fft-type questions

Warm-up Exercise: Compute the reduced row echelon form of

'til 10:57

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{bmatrix}$$

for later.

page 2, Wed notes

Theorem: Let $A_{n \times n}$ be a square matrix. Then A has an inverse matrix if and only if its reduced row echelon form is the identity. In this case the algorithm illustrated in our examples will yield A^{-1}.

explanation: By the theorem, we discussed on Monday, when A^{-1} exists, the linear systems

$$A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$$

always have unique solutions ($\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$). From our previous discussions about reduced row echelon form, we know that for square matrices, solutions to such linear systems exist and are unique if and only if the reduced row echelon form of A is the identity matrix. Thus by logic, whenever A^{-1} exists, A reduces to the identity.

In this case that A does reduce to I, we search for A^{-1} as the solution matrix X to the matrix equation

$$A\,X = I$$

i.e.

$$A \left[\begin{array}{c|c|c|c} col_1(X) & col_2(X) & \ldots & col_n(X) \end{array} \right] = \left[\begin{array}{c|c|c|c} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & & 1 \end{array} \right]$$

Because A reduces to the identity matrix, we may solve for X column by column as in the examples we've worked, by using a chain of elementary row operations:

$$[\,A \mid I\,] \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow [\,I \mid B\,],$$

and deduce that the columns of X are exactly the columns of B, i.e. $X = B$. Thus we know that

$$A\,B = I\,.$$

To realize that $B\,A = I$ as well, we would try to solve $B\,Y = I$ for Y, and hope $Y = A$. But we can actually verify this fact by reordering the columns of $[\,I \mid B\,]$ to read $[\,B \mid I\,]$ and then reversing each of the elementary row operations in the first computation, i.e. create the reversed chain of elementary row operations,

$$[\,B \mid I\,] \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow [\,I \mid A\,]\,.$$

so $B\,A = I$ also holds. (This is one of those rare times when matrix multiplication actually is commuative.)

To summarize: If A^{-1} exists, then solutions $\underline{\boldsymbol{x}}$ to $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ always exist and are unique, so the reduced row echelon form of A is the identity. If the reduced row echelon form of A is the identity, then A^{-1} exists, because we can find it using the algorithm above. That's exactly what the Theorem claims.

Saying the same thing in lots of different ways (important because it ties a lot of our Chapter 1-2 ideas together): Can you explain why these are all equivalent?

<u>The invertible matrix theorem</u> (page 114) great review of everything

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a) A is an invertible matrix.

$\Updownarrow$ did this

b) The reduced row echelon form of A is the $n \times n$ identity matrix.

$\Updownarrow$

c) A has n pivot positions

d) The equation $A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$ has only the trivial solution $\underline{\mathbf{x}} = \underline{\mathbf{0}}$.

e) The columns of A form a linearly independent set.

f) The linear transformation $T(\underline{\mathbf{x}}) := A\,\underline{\mathbf{x}}$ is one-one.

g) The equation $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ has at least one solution for each $\underline{\boldsymbol{b}} \in \mathbb{R}^n$.

h) The columns of A span $\mathbb{R}^n$.

i) The linear transformation $T(\underline{\mathbf{x}}) := A\,\underline{\mathbf{x}}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j) There is an $n \times n$ matrix C such that $C\,A = I$.

k) There is an $n \times n$ matrix D such that $A\,\mathrm{D} = I$.

l) A^T is an invertible matrix.

Wed Feb 7

- 2.2-2.3 Matrix inverses: the product of elementary matrices approach to matrix inverses

Announcements:

- on exam, you'll need to know basic defintions (closed book, closed note)
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- today: massive thm to review some of material
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Warm-up Exercise: Definitions: e.g.

a) $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ are linearly independent means

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0} \implies c_1 = c_2 = \ldots = c_n = 0$$

b) $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\} = \{x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n$, such that each $x_j \in \mathbb{R}$, $j = 1, 2, \ldots n\}$

c) $T: \mathbb{R}^n \to \mathbb{R}^m$ linear transformation

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \forall\, \vec{u}, \vec{v} \in \mathbb{R}^n$$

$$T(c\vec{u}) = cT(\vec{u}) \quad \forall\, \vec{u} \in \mathbb{R}^n,\ c \in \mathbb{R}$$

*

- we showed that matrix transformations are linear: $T(\vec{x}) = A\vec{x}$

$$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$$
$$A(c\vec{x}) = cA\vec{x}$$

- we showed that these linear transformations are matrix trans.

Exercise 1) Show that if A, B, C are invertible matrices, then

$$(A\,B)^{-1} = B^{-1}\,A^{-1}.$$
$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

the matrix for T is given by

$$\left[\begin{array}{c:c:c:c} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{array}\right]$$

Theorem The product of $n \times n$ invertible matrices is invertible, and the inverse of the product is the product of their inverses in reverse order.

Saying the same thing in lots of different ways (important because it ties a lot of our Chapter 1-2 ideas together): Can you explain why these are all equivalent?

The invertible matrix theorem (page 114)

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a) A is an invertible matrix.

b) The reduced row echelon form of A is the $n \times n$ identity matrix.

c) A has n pivot positions

$a \Rightarrow b$ if A^{-1} exists, then the solution $\vec{x}$ to $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$

$b \Leftrightarrow c$

so the solution $\vec{x}$ is unique

$b \Rightarrow a$ via our algorithm. (augment A with I, reduce)

• since each such eqtn has a solution there's a pivot in each row

n pivots same as $\text{rref}(A) = I$

(• or, solutions are unique so no free variables, so each column has a pivot so n pivots)

d) The equation $A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$ has only the trivial solution $\underline{\mathbf{x}} = \underline{\mathbf{0}}$.

write

$A = [\vec{a}_1, \vec{a}_2, \ldots \vec{a}_n]$

e) The columns of A form a linearly independent set.

f) The linear transformation $T(\underline{\mathbf{x}}) := A\,\underline{\mathbf{x}}$ is one-one.

$d \Longrightarrow e.$ want to show that

(1) $c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n = \vec{0}$ then $c_1 = c_2 = \cdots = c_n = 0$

$$\|$$

(2) $A\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$

if d is true $\vec{c} = \vec{0}$, so $c_1 = c_2 = \cdots = c_n = 0$

$e \Rightarrow d$: just read previous paragraph backwards : If e is true then (1) is true, so the only sol'n to (2) is $\vec{c} = \vec{0}$.

$d \Rightarrow f$ $T(\vec{x})$ one-one means the problem $T(\vec{x}) = \vec{b}$ always has unique solutions.

if $A\vec{x} = \vec{b}$ & $A\vec{y} = \vec{b}$ ~~and~~ is $\vec{x} = \vec{y}$?

$A\vec{x} - A\vec{y} = \vec{0}$

$A(\vec{x} - \vec{y}) = \vec{0}$ if (d) is true, $\vec{x} - \vec{y} = \vec{0}$

$\vec{x} = \vec{y}$ ▪

$f \Rightarrow d.$

if sol'tns to $A\vec{x} = \vec{b}$ are unique, then the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$!

connect abc, def

$a \Rightarrow d$: if A^{-1} exists the solution to $A\vec{x} = \vec{0}$

$\vec{x} = A^{-1}\vec{0} = \vec{0}$

$d \Rightarrow b$: no free parameter $\Longrightarrow$ rref$(A) = I$

g) The equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ has at least one solution for each $\underline{\boldsymbol{b}} \in \mathbb{R}^n$.

h) The columns of A span $\mathbb{R}^n$.

i) The linear transformation $T(\underline{\boldsymbol{x}}) := A\,\underline{\boldsymbol{x}}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

g), h), i) are saying exactly the same thing, but in different contexts.

h) Write $A = [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]$ in terms of its columns.

$$\text{span}\{\vec{a}_1, \vec{a}_2, \dots \vec{a}_n\} = \left\{ x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n \text{ such that } \begin{matrix} x_j \in \mathbb{R} \\ 1 \le j \le n \end{matrix} \right\}$$

$$= \{ A\vec{x} \text{ such that } \vec{x} \in \mathbb{R}^n \}$$

so, saying each $\vec{b} \in \mathbb{R}^n$ is in the span of the columns of A is saying the equation $A\vec{x} = \vec{b}$ always has at least one solution

So, $g \Leftrightarrow h$.

i) a function $f: X \to Y$ is <u>onto</u> means the equation $f(x) = b$ always has at least one solution x, for each $b \in Y$.

So asking whether $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto, is asking whether the equation $A\vec{x} = \vec{b}$ always has at least one solution

So $g \Leftrightarrow i$

connect g, h, i to a, b, c:

$b \Rightarrow g$: iff $\text{rref}(A) = I$ then the eqtn $A\vec{x} = \vec{b}$ always has a solution, because rref(A) has a pivot in each row.

$g \Rightarrow b$: If $A\vec{x} = \vec{b}$ always has at least one sol'n, rref(A) has a pivot in each row, so since A is a square matrix, $\text{rref}(A) = I$

j) There is an $n \times n$ matrix C such that $C A = I$.

k) There is an $n \times n$ matrix D such that A D $= I$.

l) A^T is an invertible matrix.

a) $\Rightarrow$ j, k : if A^{-1} exists, then let $C = A^{-1}$ for j)
let $D = A^{-1}$ for k)

$a \Rightarrow l$, recall that $(AB)^T = B^T A^T$
so, if A has an inverse, A^{-1} is the matrix
B for which
$AB = I \quad \& \quad BA = I$

Therefore $(AB)^T = I^T \quad ; \quad (BA)^T = I^T$
$B^T A^T = I \quad ; \quad A^T B^T = I$

So A^T has an inverse, given by $(A^T)^{-1} = (A^{-1})^T$

$j \Rightarrow a)$:
if $CA = I$
then the only solution $\vec{x}$ to $A\vec{x} = \vec{0}$ is $\vec{x} = 0$, since

$$A\vec{x} = \vec{0}$$
$$\Rightarrow CA\vec{x} = C\vec{0} = \vec{0}$$
$$\Rightarrow I\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}.$$

So, $j \Rightarrow d \Rightarrow a$

$k \Rightarrow a)$
If $AD = I$
then $AD\vec{b} = I\vec{b} = \vec{b}$ for any $\vec{b} \in \mathbb{R}^n$.
so the eqtn $A\vec{x} = \vec{b}$ has a solution $\vec{x} = D\vec{b}$.

So $k \Rightarrow g \Rightarrow a$

$l \Rightarrow a)$: If $(A^T)^{-1}$ exists, then apply $a \Rightarrow l$ above to the matrix A^T to see that $(A^T)^T = A$ has an inverse

$\boxed{///}$

Wed Feb 7

- 2.2-2.3 Matrix inverses: the product of elementary matrices approach to matrix inverses

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a) $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ are linearly independent means

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b) $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\} = \{x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n$, such that each $x_j \in \mathbb{R}$, $j = 1, 2, \ldots n\}$

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*

- we showed that matrix transformations are linear: $T(\vec{x}) = A\vec{x}$
  $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$
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- we showed that these linear transformations are matrix trans.

the matrix for T
is given by

$$\left[\begin{array}{c|c|c|c} T(\vec{e}_1) & T(\vec{e}_2) & & T(\vec{e}_n) \end{array} \right]$$

<u>Exercise 1)</u> Show that if A, B, C are invertible matrices, then

$$(A\,B)^{-1} = B^{-1}\,A^{-1}.$$
$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

<u>Theorem</u> The product of $n \times n$ invertible matrices is invertible, and the inverse of the product is the product of their inverses in reverse order.

Our algorithm for finding the inverse of a matrix can be reformulated in terms of a product of so-called "elementary" matrices. This product idea will pay off elsewhere. To get started, let's notice an analog of the fact that a matrix times a vector is a linear combination of the matrix columns. That was in fact how we defined matrix times vector in week 2, although we usually use the dot product way of computing the product entry by entry, instead:

<u>Definition</u> (from 1.4) If A is an $m \times n$ matrix, with columns $\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n$ (in $\mathbb{R}^m$) and if $\underline{\boldsymbol{x}} \in \mathbb{R}^n$, then $A\,\underline{\boldsymbol{x}}$ is defined to be the linear combination of the columns, with weights given by the corresponding entries of $\underline{\boldsymbol{x}}$. In other words,

$$A\,\underline{\boldsymbol{x}} = \left[\underline{\boldsymbol{a}}_1 \;\; \underline{\boldsymbol{a}}_2 \;\; \ldots \underline{\boldsymbol{a}}_n\right]\underline{\boldsymbol{x}} := x_1\,\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \;\ldots x_n\,\underline{\boldsymbol{a}}_n \,.$$

<u>Theorem</u> If we multiply a *row vector* times an $n \times m$ matrix B we get a linear combination of the *rows* of B: <u>proof:</u> We want to check whether

$$\underline{\boldsymbol{x}}^T B = \left[\; x_1 \;\; x_2 \;\; \cdots \;\; x_n \;\right] \begin{bmatrix} \underline{\boldsymbol{b}}_1 \\ \underline{\boldsymbol{b}}_1 \\ \vdots \\ \underline{\boldsymbol{b}}_n \end{bmatrix} = x_1\,\underline{\boldsymbol{b}}_1 + x_2\,\underline{\boldsymbol{b}}_2 + \;\ldots x_n\,\underline{\boldsymbol{b}}_n \,.$$

where the rows of B are given by the row vectors $\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n$. This proposed identity is true if and only if its transpose is a true identity. But the transpose of the left side is

$$\left(\underline{\boldsymbol{x}}^T B\right)^T = B^T \left(\underline{x}^T\right)^T = B^T\,\underline{\boldsymbol{x}}$$

$$= \left[\underline{\boldsymbol{b}}_1^T \;\underline{\boldsymbol{b}}_2^T \;\ldots \underline{\boldsymbol{b}}_n^T\right]\underline{\boldsymbol{x}}$$

$$x_1\,\underline{\boldsymbol{b}}_1^T + x_2\,\underline{\boldsymbol{b}}_2^T + \;\ldots x_n\,\underline{\boldsymbol{b}}_n^T$$

which is the transpose of the right side of the proposed identity. So the identity is true.

Q.E.D.

Exercise 2a Use the Theorem on the previous page and work row by row on so-called "elementary matrix" E_1 on the right of the product below, to show that $E_1\,A$ is the result of replacing $row_3(A)$ with $row_3(A) - 2\,row_1(A)$, and leaving the other rows unchanged:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} =$$

2b) The inverse of E_1 must undo the original elementary row operation, so must replace any $row_3(A)$ with $row_3(A) + 2\,row_1(A)$. So it must be true that

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Check!

2c) What 3×3 matrix E_2 can we multiply times A, in order to multiply $row_2(A)$ by 5 and leave the other rows unchanged. What is E_2^{-1} ?

2d) What 3×3 matrix E_3 can we multiply time A, in order to swap $row_1(A)$ with $row_3(A)$? What is E_3^{-1} ?

Definition An *elementary matrix* E is one that is obtained by doing a single elementary row operation on the identity matrix.

Theorem Let $E_{m \times m}$ be an elementary matrix. Let $A_{m \times n}$. Then the product $E\,A$ is the result of doing the same elementary row operation to A that was used to construct E from the identity matrix.

Algorithm for finding A^{-1} re-interpreted: Suppose a sequence of elementary row operations reduces the $n \times n$ square matrix A to the identity I_n. Let the corresponding elementary matrices, in order, be given by

$$E_1, E_2\,, \ldots\ \ E_p.$$

Then we have

$$E_p\left(E_{p-1} \ldots . E_2\left(E_1(A)\right) \ldots\right) = I_n$$

$$E_p\,E_{p-1} \ldots . E_2\,E_1\,A = I_n\,.$$

So,

$$A^{-1} = E_p\,E_{p-1} \ldots . E_2\,E_1.$$

Notice that

$$E_p\,E_{p-1} \ldots . E_2\,E_1 = E_p\,E_{p-1} \ldots . E_2\,E_1\,I_n$$

so we have obtained $A^{-1}$by starting with the identity matrix, and doing the same elementary row operations to it that we did to A, in order to reduce A to the identity. I find this explanation of our original algorithm to be somewhat convoluted, but as I said, the matrix product decomposition idea is going to pay dividends elsewhere.

Also, notice that we have ended up "factoring" A into a product of elementary matrices:

$$A = \left(A^{-1}\right)^{-1} = \left(E_p\,E_{p-1} \ldots . E_2\,E_1\right)^{-1} = E_1^{-1}\,E_2^{-1} \ldots . E_{p-1}^{-1}\,E_p^{-1}\ \ .$$