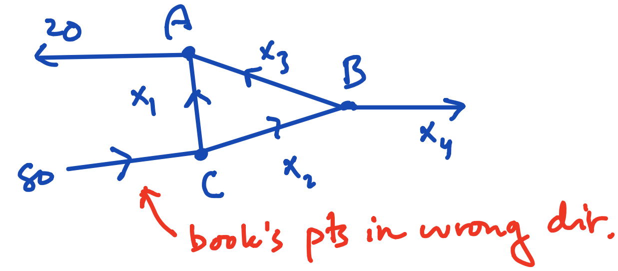
Math 2270-004 Week 3 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an outline of what we plan to cover. These notes cover material in 1.5-1.8.

Mon Jan 22

- 1.5-1.6 review of facts we know, and some applications of systems of linear equations.

Announcements: wrong arrows on 1.6.11 :



Warm-up Exercise: Vocabulary "quiz"! Define 'til 12:58

1. a linear combination of $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$
is any $\vec{v} = x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n$ with $x_1, x_2 \ldots x_n \in \mathbb{R}$

2. span $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$
= collection of all linear combinations

geometrically span$\{\vec{v}_1\}$ is line thru $\vec{0}$. $(\vec{v}_1 \neq \vec{0})$
span$\{\vec{v}_1, \vec{v}_2\}$ is a plane thru $\vec{0}$ (as long as $\vec{v}_1, \vec{v}_2$ not parallel)

3. $A\vec{x}$, for $A_{\underline{m \times n}}$ & $\vec{x} \in \mathbb{R}^n$

$$\left[\vec{a}_1 \middle| \vec{a}_2 \middle| \cdots \vec{a}_n\right]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a} + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n$$

(also $\begin{bmatrix} Row_1(A) \\ \hline Row_2(A) \\ \hline \vdots \\ Row_m(A) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} Row_1(A)\cdot\vec{x} \\ Row_2(A)\cdot\vec{x} \\ \vdots \\ Row_m(A)\cdot\vec{x} \end{bmatrix}$)

4. Homogeneous matrix equation
$A\vec{x} = \vec{0}$
Nonhomogeneous " "
$A\vec{x} = \vec{b}$, $\vec{b} \neq \vec{0}$.

5. Pivot location for a matrix B
location of a pivot in r.r.e.f.(B).

6. 4 conditions for a matrix to be in reduced row echelon form
(1) all zero rows are at bottom
(2) pivots (the 1st non-zero entry in a row, move to the right as you descend rows
(3) pivots = 1
(4) entries in any pivot column = 0, except for pivot

Review and consolidation of facts from sections 1.1-1.5:

1) If $A_{m \times n} = [\underline{\boldsymbol{a}}_1\ \underline{\boldsymbol{a}}_2\ \ldots\ \underline{\boldsymbol{a}}_n]$ is expressed in terms of its columns, with a_{ij} being the i^{th} entry of $\underline{\boldsymbol{a}}_j$ then we know

$$A\,\underline{\boldsymbol{x}} := x_1\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \ldots + x_n\,\underline{\boldsymbol{a}}_n = \begin{bmatrix} x_1a_{11} + x_2a_{12} + \ldots + x_na_{1n} \\ x_1a_{21} + x_2a_{22} + \ldots + x_na_{2n} \\ \vdots \\ x_1a_{m1} + x_2a_{m2} + \ldots + x_na_{mn} \end{bmatrix} = \begin{bmatrix} Row_1(A)\cdot\underline{\boldsymbol{x}} \\ Row_2(A)\cdot\underline{\boldsymbol{x}} \\ \vdots \\ Row_m(A)\cdot\underline{\boldsymbol{x}} \end{bmatrix}.$$

So the matrix equation

$$A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$$

from 1.4 represents

1a) systems of linear equations, as in 1.1-1.2, as well as

1b) vector (linear combination) equations, as in section 1.3.

The solution set in any such problem is found and understood by reducing the augmented matrix $[\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n, \underline{\boldsymbol{b}}]$ to see if the system is consistent, and then backsolving when it is.

2) We can understand a lot about the geometry of the solution set of the matrix equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ based on the shape of the reduced row echelon form of the augmented matrix

$$[A, \underline{\boldsymbol{b}}] = \left[\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \dots \underline{\boldsymbol{a}}_n, \underline{\boldsymbol{b}}\right],$$

or often just on the shape of the reduced row echelon form of

$$A = \left[\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \dots \underline{\boldsymbol{a}}_n\right]$$

alone.

2a) The system is inconsistent if and only if what is true about $rref([A, \underline{\boldsymbol{b}}])$?

$\exists$ row "0" on left, 1 on right

i.e. right-most column has a pivot (eqtn says $0=1$)

row: $0\ 0\ 0\ 0 \mid 1$

2b) If the system is consistent then there is a ~~unique~~ exactly one solution $\underline{\boldsymbol{x}}$ to $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ if and only if what is true about $rref(A)$?

every column has a pivot

$$\left[\begin{array}{cc|c} 1 & 0 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & 0 \end{array}\right] \quad \text{unique}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & c_1 \\ 0 & 0 & 1 & c_2 \end{array}\right] \quad \text{not unique.}$$

2c) If the system is consistent then the number of free variables in the solution is given by what number related to $rref(A)$?

# of cols of rref(A) without pivots.

2d) For a fixed matrix A the matrix equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ is consistent for all possible choices of $\underline{\boldsymbol{b}}$ if and only if what is true about $rref(A)$?

no zero rows in rref(A)

i.e. every row in rref(A) has a pivot.

because I can solve for the pivot variable in terms of the later variables (that shows $A\vec{x} = \vec{b}$ is always consistent in this case).

e.g.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & d_1 \\ 0 & 1 & 2 & d_2 \end{array}\right]$$

$A\vec{x} = \vec{b}$

Note: if instead, rref(A) has a row of zeroes: $A \xrightarrow{rref} \begin{matrix} \ast\ast\ast\ast \\ 0\,0\,0\,0 \end{matrix}$

then there are inconsistent systems $A\vec{x} = \vec{b}$: $A \mid \vec{b} \xleftarrow{rref^{-1}} \left[\begin{array}{c|c} \ast\ast\ast\ast & \vdots \\ 0\,0\,0\,0 & 1 \end{array}\right]$

3) Let $A_{n \times n}$ be a square matrix.

3a) Then the matrix equation $A\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ is consistent for all possible choices of $\underline{\boldsymbol{b}}$ if and only if what is true about $rref(A)$?

every row of rref(A) must have a pivot (=1), so n pivots

i.e. $rref(A) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & \cdots 0 & 1 \end{bmatrix}$

$:= I$

3b) Then solutions to the matrix equation $A\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ are unique if and only if what is true about $rref(A)$?

every column of rref(A) must have a pivot, i.e. n pivots

i.e. $rref(A) = I$ also

4) spanning sets

4a) Fewer than m vectors in $\mathbb{R}^m$, $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n\}$ with $n < m$, will never span all of $\mathbb{R}^m$ because

2 vectors in $\mathbb{R}^3$: $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$ solvable for all $\vec{b} \in \mathbb{R}^3$?

$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}_{3\times 2} \xrightarrow{rref} \begin{bmatrix} * & * \\ * & * \\ 0 & 0 \end{bmatrix}$

so can't always solve $A\vec{x} = \vec{b}$.

(this same reasoning holds whenever $n<m$.)

4b) Exactly n vectors in $\mathbb{R}^n$, $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n\}$ span $\mathbb{R}^n$ if and only if

i.e. can always solve $A\vec{x} = \vec{b}$ for $\vec{x}$

every row has a pivot, i.e. as in 3a),

$rref(A) = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I$ (as in 3a)

5) homogeneous and nonhomogeneous systems

5a) The *homogeneous* matrix equation, $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$ is always consistent.

5b) If $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ is also consistent, then the solution set to $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ is a translation of the solution set to $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$. In other words, the solution set of $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ is the set of all vectors

$$\underline{\boldsymbol{w}} = \underline{\boldsymbol{p}} + \underline{\boldsymbol{v}}_h$$

where $\underline{\boldsymbol{p}}$ is a particular solution to $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ and $\underline{\boldsymbol{v}}_h$ is any solution of the homogeneous equation

$$A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}.$$

Tuesday warmup, from Monday's notes

Exercise ~~1)~~ I gave an abstract explanation for why 5b is true on Friday. We can see it more concretely if we understand how this exercise below generalizes: A double augmented matrix for $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$ and $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ and its reduced row echelon form are shown. Find and express the homogeneous and non-homogeneous solutions in linear combination form.

'til 12:58

$$\left[\begin{array}{cccc|c:c} 2 & -1 & 1 & 5 & 0 & 1 \\ -3 & 2 & -1 & -8 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & 2 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c:c} 1 & 0 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

A $\qquad$ $\vec{0} \,\vdots\, \vec{b}$ $\qquad$ x_3 free $\quad$ x_4 free

$$A\vec{x} = \vec{0}: \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - 2x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

$$A\vec{x} = \vec{b}: \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 - x_3 - 2x_4 \\ 3 - x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \underbrace{x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}}$$

$$= \vec{p} + \vec{v}_H$$

1.6 Some applications of matrix equations.

Exercise 2) Balance the following chemical reaction equation, for the burning of propane:

$$\overset{1}{\overset{\shortparallel}{x_1}}\, C_3H_8 + \overset{5}{\overset{\shortparallel}{x_2}}\, O_2 \rightarrow \overset{3}{\overset{\shortparallel}{x_3}}\, CO_2 + \overset{4}{\overset{\shortparallel}{x_4}}\, H_2O$$

$$\begin{array}{l} C \rightarrow \\ H \rightarrow \\ O \rightarrow \end{array} \quad x_1\begin{bmatrix}3\\8\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\2\end{bmatrix} = x_3\begin{bmatrix}1\\0\\2\end{bmatrix} + x_4\begin{bmatrix}0\\2\\1\end{bmatrix}$$

$$x_1\begin{bmatrix}3\\8\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\2\end{bmatrix} - x_3\begin{bmatrix}1\\0\\2\end{bmatrix} - x_4\begin{bmatrix}0\\2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}3 & 0 & -1 & 0\\ 8 & 0 & 0 & -2\\ 0 & 2 & -2 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Hint: after you set up the problem, the following reduced row echelon form computation will be helpful:

$$\left[\begin{array}{cccc|c} 3 & 0 & -1 & 0 & 0\\ 8 & 0 & 0 & -2 & 0\\ 0 & 2 & -2 & -1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{4} & 0\\ 0 & 1 & 0 & -\frac{5}{4} & 0\\ 0 & 0 & 1 & -\frac{3}{4} & 0 \end{array}\right]$$

$x_1 = \frac{1}{4}x_4$

$x_2 = \frac{5}{4}x_4$

$x_3 = \frac{3}{4}x_4$

$x_4 =$ free

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = x_4\begin{bmatrix}\frac{1}{4}\\ \frac{5}{4}\\ \frac{3}{4}\\ 1\end{bmatrix} \qquad \text{Let } x_4 = 4: \quad \begin{bmatrix}1\\5\\3\\4\end{bmatrix}$$

Exercise 3) Consider the following traffic flow problem (from our text): What are the possible flow patterns, based on the given information and that the streets are one-way, so none of the flow numbers can be negative?

EXAMPLE 2 The network in Figure 2 shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.

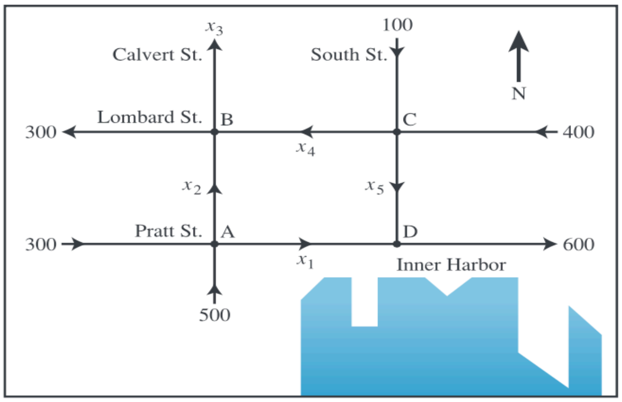


FIGURE 2 Baltimore streets.

@A: flow in = flow out

$$800 = x_1 + x_2$$

B: $x_2 + x_4 = 300 + x_3$

C: $500 = x_4 + x_5$

D: $x_1 + x_5 = 600.$

$$\begin{aligned} x_1 + x_2 \quad &= 800 \\ x_2 - x_3 + x_4 &= 300 \\ x_4 + x_5 &= 500 \\ x_1 \quad + x_5 &= 600 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \end{array}\right]$$

Hint: If you set up the flow equations for intersections *A*, *B*, *C*, D in that order, the following reduced row echelon form computation may be helpful:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{array}\right]$$

$x_1 = 600 - x_5$

$x_2 = 200 + x_5$

$x_3 = 400$

$x_4 = 500 - x_5$

$x_5 =$ free

Tues Jan 23

- 1.7 linear dependence and independence. Connections to reduced row echelon form.

<u>Announcements:</u>

1.7 1, 5, 7 (15, 17, 27, 28, 31, 39, 42)

- finish Monday's notes : 1.6 applications
- start §1.7 on linear independence
  (continue tomorrow)

<u>Warm-up Exercise</u>: (was Exercise 1 in Monday's notes)

1.7 When we are discussing the span of a collection of vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ we would like to know that we are being efficient in describing this span, and not wasting any free parameters because of redundancies in the vectors. For example, the most efficient way to describe a plane in $\mathbb{R}^3$ is as the span of exactly two vectors, rather than as the span of three or more. This has to do with the concept of "linear independence":

Definition:

not dependent

a) An indexed set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ is said to be linearly independent if no one (1) of the vectors is a linear combination of (some) of the other vectors. The concise way to say this is that the only way $\underline{\mathbf{0}}$ can be expressed as a linear combination of these vectors,

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n = \underline{\mathbf{0}},$$

is for all of the weights $c_1 = c_2 = \ldots = c_n = 0$.

start

b) An indexed set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ is said to be linearly dependent if (1) at least one of these vectors *is* a linear combination of (some) of the other vectors. The concise way to say this is that there *is* some way to write $\underline{\mathbf{0}}$ as a linear combination of these vectors

(2)

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n = \underline{\mathbf{0}}$$

where *not all* of the $c_j = 0$. (We call such an equation a linear dependency. Note that if we have any such linear dependency, then any $\underline{\boldsymbol{v}}_j$ with $c_j \neq 0$ is a linear combination of the remaining $\underline{\boldsymbol{v}}_k$ with $k \neq j$. We say that such a $\underline{\boldsymbol{v}}_j$ is linearly dependent on the remaining $\underline{\boldsymbol{v}}_k$.)

(1) means some $\vec{v}_j = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n$ (no $d_j\vec{v}_j$ term) not all d_j's are zero

$\Downarrow$

so $d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n - \vec{v}_j = \vec{0}$ i.e. (2) holds.

(2) If $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$ with not all weights $= 0$

assume $c_j \neq 0$ (pick j)

then $c_j\vec{v}_j = \underbrace{-c_1\vec{v}_1 - c_2\vec{v}_2 - \cdots - c_n\vec{v}_n}_{\text{no } \vec{v}_j \text{ term.}}$

$$\vec{v}_j = -\frac{c_1}{c_j}\vec{v}_1 - \frac{c_2}{c_j}\vec{v}_2 - \cdots - \frac{c_n}{c_j}\vec{v}_n$$

Note: The only set of a single vector $\{\underline{\boldsymbol{v}}_1\}$ that is dependent is if $\underline{\boldsymbol{v}}_1 = \underline{\boldsymbol{0}}$. The only sets of two non-zero vectors, $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2\}$ that are linearly dependent are when one of the vectors is a scalar multiple of the other one. For more than two vectors the situation is more complicated.

dependency eqtn $c_1\vec{v}_1 = \vec{0}$ is there a soltn $c_1 \neq 0$

$$c_1 \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

if any $a_j \neq 0$ then $c_1 = 0$

so if $\vec{v} \neq \vec{0}$, $\{\vec{v}\}$ is indep.

Exercise 1a) Is this set linearly dependent or independent? $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$?

$$c_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ for $c_1 \neq 0$ YES

so $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$ is dependent

1b) Is this set of vectors linearly dependent or independent? $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \end{bmatrix}\right\}$?

$\vec{v}_1$ $\vec{v}_2$

$$\vec{v}_2 = -3\vec{v}_1$$

(or $3\vec{v}_1 + \vec{v}_2 = \vec{0}$)

Example

The set of vectors $\left\{ \underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \underline{\boldsymbol{v}}_3 = \begin{bmatrix} -2 \\ 8 \end{bmatrix} \right\}$ in $\mathbb{R}^2$ is linearly dependent because, as we showed when we were introducing vector equations (and as we can quickly recheck),

$$-3.5\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1.5\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

① $\vec{v}_3$ is linearly dependent on $\vec{v}_1$ & $\vec{v}_2$

We can also write this linear dependency as

$$-3.5\underline{\boldsymbol{v}}_1 + 1.5\underline{\boldsymbol{v}}_2 - \underline{\boldsymbol{v}}_3 = \underline{\mathbf{0}}$$

② 2^{nd} way to say $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are dependent

(or any non-zero multiple of that equation.)

Exercise 2) For a linearly independent set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$, every vector $\underline{\boldsymbol{v}}$ in their span can be written as $\underline{\boldsymbol{v}} = d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + \ldots + d_n\underline{\boldsymbol{v}}_n$ uniquely, i.e. for exactly one choice of weights $d_1, d_2, \ldots d_n$. This is not true if vectors are dependent. Explain these facts. (You can illustrate these facts with the vectors in the example above.)

Note that the set of vectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n\}$ is linearly dependent if and only if there are non-zero solutions $\underline{c}$ to the homogeneous matrix equation

$$A\,\underline{c} = \underline{\mathbf{0}}$$

for the matrix $A = [\underline{v}_1\ \underline{v}_2\ \ldots \underline{v}_n]$ having the given vectors as columns. Thus all linear independence/dependence questions can be answered using reduced row echelon form.

WEDNESDAY WARM-UP 'til 12:58

Exercise 3) Show that the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ \underline{v}_3 = \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$$

~~are linearly dependent (even though no two of them are~~ scalar multiples of each other). What does this mean geometrically about the span of these three vectors? Find all dependencies

they only span a plane in $\mathbb{R}^3$.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{dependency equation}$$

is equivalent

secretly a homogeneous solution space problem

$$\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \vec{v}_3\right] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hint: You might find this computation useful: Find all dependencies.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & 6 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

backsolve for
$c_1 = -2c_3 = -2t$
$c_2 = -3c_3 = -3t$
$c_3 = \text{free} = t$

one "basic" dependency
$c_1 = -2$
$c_2 = -3$
$c_3 = 1$

$$-2\underbrace{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}_{\vec{v}_1} - 3\underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\vec{v}_2} + \underbrace{\begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}}_{\vec{v}_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{\text{all dependencies:}}$ $\boxed{\vec{c} = t\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix},\ t \in \mathbb{R}}$ ✓

Exercise 4) Are the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ \underline{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

linearly independent or dependent? Hint:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]. \Rightarrow \begin{matrix} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{matrix}$$

independent

dependency eqtn is

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 5a) Why must more than three vectors in $\mathbb{R}^3$ be linearly dependent?

e.g. 4:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$$

$$3\left\{ \underbrace{\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \vec{v}_3 \middle| \vec{v}_4\right]}_{4} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right.$$

- at least 1 column is not a pivot column.
- The system is consistent
- at least 1 free parameter so ∞'ly many dependencies,

5b) How about more than m vectors in $\mathbb{R}^m$?

n vectors in $\mathbb{R}^m$, $n > m$

\# free parameters in dependencies is at least $n-m$.

5c) If you are given a set of exactly n vectors in $\mathbb{R}^n$ how can you check whether or not they are linearly independent? How does your criterion compare to the condition that will guarantee that the n vectors span $\mathbb{R}^n$?

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$$

$$n\left\{ \underbrace{\left[\vec{v}_1 \vdots \vec{v}_2 \vdots \cdots \vec{v}_n\right]}_{n} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right.$$

$= $ "A"

for $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ to be linearly ind. $\operatorname{rref}(A) = I$ (because each column has a pivot)

how does that compare to condition that $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\} = \mathbb{R}^n$?

no zero rows in $\operatorname{rref}(A)$

$\Rightarrow n$ pivots $\Rightarrow \operatorname{rref}(A) = I$

SAME CONDITION

if n vectors in $\mathbb{R}^n$ span $\mathbb{R}^n$ & are linearly independent, we call them a "basis" for $\mathbb{R}^n$

Wed Jan 24

- 1.7 Linear dependence/independence continued, and why each matrix has a unique reduced row echelon form

Announcements:
- quiz today
- finish Tuesday's & today's notes

moral of §1.7 : linear independence/dependence questions are "the same as" homogeneous matrix equation questions

Warm-up Exercise: Exercise 3 in Tuesday's notes dependency ?'s are homogeneous matrix eqtn ?'s

Exercise 1) Consider the homogeneous matrix equation $A\,\underline{\boldsymbol{x}} = \underline{\mathbf{0}}$, with the matrix A (and its reduced row echelon form) shown below:

$$A := \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 1 & 0 \\ 2 & 4 & 1 & 4 & 1 & 0 \\ -2 & -4 & 0 & -2 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$\vec{v}_1 \quad \vec{v}_2 \quad \ldots \quad \vec{v}_5$

Find and express the solutions to this system in linear combination form. Note that you are finding all of the dependencies for the collection of vectors that are the columns of A, namely the set

$$\left\{ \underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \underline{\boldsymbol{v}}_1 = \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}, \underline{\boldsymbol{v}}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \underline{\boldsymbol{v}}_4 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}, \underline{\boldsymbol{v}}_5 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

$c_1 = -2p - r - t$
$c_2 = \text{free} = p$
$c_3 = -2r + t$
$c_4 = r \quad$ free
$c_5 = t \quad$ free

$$\vec{c} = p \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$ all dependencies

e.g. $p = 1, \; r = t = 0$

$$-2 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix} + \vec{0} + \vec{0} + \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2\vec{v}_1 + \vec{v}_2 = \vec{0}$.

Exercise 2) Focus on the idea that solutions to homogeneous matrix equations correspond exactly to linear dependencies between the columns of the matrix. Now, since the solution set to a homogeneous linear system does not change as you do elementary row operations to the augmented matrix, column dependencies also do not change. Therefore the vectors that span the space of homogeneous solutions in Exercise 1 are encoding the key column dependencies in $\mathbb{R}^3$, for both the original matrix, and for the reduced row echelon form.

Check this, by reading off "easy" column dependencies in the reduced matrix; seeing that they are also dependencies in the original matrix; and that they correspond to the basis of the homogeneous solution space. Magic! We will use this magic in important interesting ways, later in the course.

$$A := \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 4 & 1 \\ -2 & -4 & 0 & -2 & -2 \end{bmatrix} \\ \begin{matrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{matrix} \end{array} \rightarrow \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 & \vec{w}_4 & \vec{w}_5 \end{matrix} \end{array}$$

$-2\vec{v}_1 + \vec{v}_2 = \vec{0}$

$\vec{v}_2 = 2\vec{v}_1 \Longleftarrow \vec{w}_2 = 2\vec{w}_1$

$\vec{v}_3$ & $\vec{v}_1$ are indep. $\Longleftarrow$ $\vec{w}_3$ & $\vec{w}_1$ are independent

$\vec{v}_4 = \vec{v}_1 + 2\vec{v}_3 \Longleftarrow \vec{w}_4 = 1\vec{w}_1 + 2\vec{w}_3$

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \checkmark$$

$\vec{v}_5 = \vec{v}_1 - \vec{v}_3 \Longleftarrow \vec{w}_5 = \vec{w}_1 - \vec{w}_3$

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \checkmark$$

Exercise 3) (This exercise explains why each matrix has only one reduced row echelon form, no matter what sequence of elementary row operations one uses to find it. We didn't have the tools to explain why this fact was true earlier in the chapter) Let $B_{4 \times 5}$ be a matrix whose columns satisfy the following dependencies:

- $col_1(B) \neq \underline{\mathbf{0}}$ (i.e. is independent)

$$col_2(B) = 3\ col_1(B)$$

$col_3(B)$ is independent of column 1

$col_4(B)$ is independent of columns 1,3.

$$col_5(B) = -3\ col_1(B) + 2\ col_3(B) - col_4(B).$$

What is the reduced row echelon form of B?

$$\Rightarrow 4\left\{\underbrace{\begin{bmatrix} 1 & 3 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{5}\right.$$

Fri Jan 26

- 1.8 introduction to linear transformations

matrix transformations.

important in CS visualization

Announcements:

Warm-up Exercise: Definitions! precise & super consise

'til 12:57

$\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ linearly independent

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0} \iff c_1 = c_2 = \ldots = c_n = 0$$

(i.e. no $\vec{v}_j$ is a linear combination of the others)

linearly dependent

$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$ for at least one choice of weights that are not all zero

(at least one $\vec{v}_j$ is a linear combo of the others)

span$\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$

all possible linear combinations of $\{\vec{v}_1, \vec{v}_2 \ldots \vec{v}_n\}$

i.e. all possible $x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n$ for $x_j \in \mathbb{R}$.

1.8 Introduction to linear transformations.

Definition: A function T which has domain equal to $\mathbb{R}^n$ and whose range lies in $\mathbb{R}^m$ is called a *linear transformation* if it transforms sums to sums, and scalar multiples to scalar multiples. Precisely, $\boxed{T : \mathbb{R}^n \to \mathbb{R}^m}$ is *linear* if and only if

name of function

domain

$$T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v}) \qquad \forall\ \underline{u}, \underline{v} \in \mathbb{R}^n$$
$$T(c\,\underline{u}) = c\,T(\underline{u}) \qquad \forall\ c \in \mathbb{R}, \underline{u} \in \mathbb{R}^n\ .$$

Notation In this case we call $\boxed{\mathbb{R}^m \text{ the } codomain.}$ We call $T(\underline{u})$ the *image of* $\underline{u}$. The *range* of T is the collection of all images $T(\underline{u})$, for $\underline{u} \in \mathbb{R}^n$.

Important connection to matrices: Each matrix $A_{m \times n}$ gives rise to a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, namely

$$T(\underline{x}) := A\,\underline{x} \qquad \forall\ \underline{x} \in \mathbb{R}^n.$$

This is because, as we checked last week,

$$A(\underline{u} + \underline{v}) = A\,\underline{u} + A\,\underline{v} \qquad \forall\ \underline{u}, \underline{v} \in \mathbb{R}^n$$
$$A(c\,\underline{u}) = c\,A\,\underline{u} \qquad \forall\ c \in \mathbb{R}, \underline{u} \in \mathbb{R}^n$$

Exercise 1) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$A_{3 \times 2}$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) := \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. = x_1 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}$$

1a Find $T(\underline{u})$ for $\underline{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. $= 2 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} - 1 \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) := \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$x_1\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \text{ or } \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

1b Is there an $\boldsymbol{x} \in \mathbb{R}^2$ whose image under T is $\boldsymbol{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$? Is $\boldsymbol{x}$ unique?

yes

$$\vec{x} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & -3 & 3 & 3 \\ 3 & 5 & 2 & 2 \\ -1 & 7 & -5 & 5 \end{array}\right]$$

1c Is $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ in the range of T?

no

inconsistent

Hints: The following reduced row echelon computation may be helpful for b, c:

$$\left[\begin{array}{cc|cc} 1 & -3 & 3 & 3 \\ 3 & 5 & 2 & 2 \\ -1 & 7 & -5 & 5 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|cc} 1 & 0 & 3/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) := \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}$$

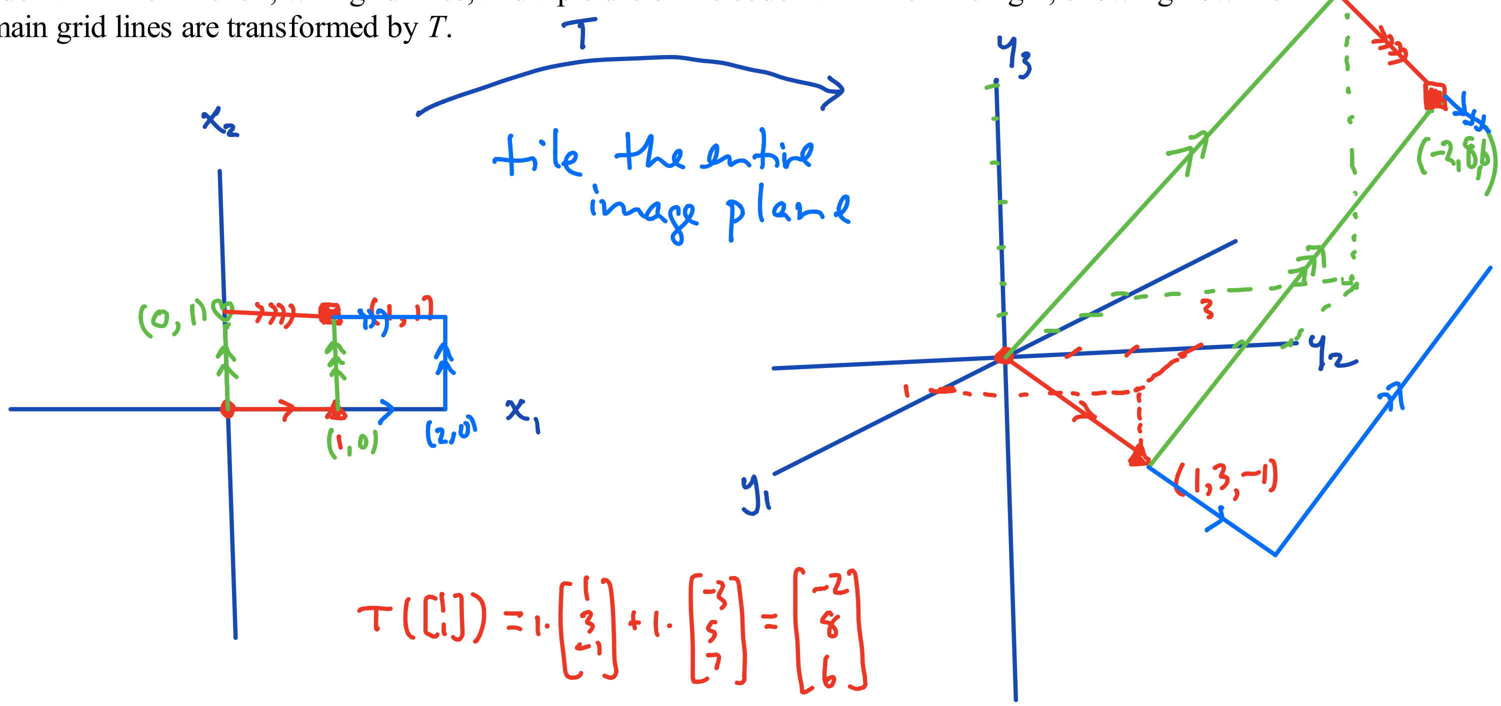
1d Describe the range of T as the span of two vectors. $= \operatorname{span}\left\{\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}\right\}$

1d2) geometrically plane in $\mathbb{R}^3$

$T\left(x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = x_2\begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}$

$T\left(x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = x_1\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ 1st col.

1e Make a geometric sketch that indicates what the transformation does. Do this by making a picture of the domain $\mathbb{R}^2$ on the left, with grid lines, and a picture of the codomain $\mathbb{R}^3$ on the right, showing how the domain grid lines are transformed by T.



$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 1\cdot\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 1\cdot\begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ 6 \end{bmatrix}$

Exercise 2) Consider the linear transformation $S : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) := \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Make a geometric sketch that indicates what the transformation does. In this case the interesting behavior is in the domain.