Math 2270-004 Week 2 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 1.3-1.5. They include material from last weeks notes that we did not get to.

Tues Jan 16

- 1.3 algebra and geometry for vector equations and linear combinations

Announcements:

Warm-up Exercise:

On Friday we defined vectors algebraically, as ordered lists of numbers. And, we defined vector addition and scalar multiplication:

<u>Definition:</u> For $\underline{\boldsymbol{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\underline{\boldsymbol{v}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$; $c \in \mathbb{R}$, then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} := \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$; $c\,\underline{\boldsymbol{u}} := \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$.

There are a number of straightforward algebra properties for vector addition and scalar multiplication:

Let $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}} \in \mathbb{R}^n$, $\quad c, d \in \mathbb{R}$. Then

(i) $\quad \underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} = \underline{\boldsymbol{v}} + \underline{\boldsymbol{u}}$

(ii) $\quad (\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) + \underline{\boldsymbol{w}} = \underline{\boldsymbol{u}} + (\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}})$

(iii) $\quad \underline{\boldsymbol{u}} + \underline{\boldsymbol{0}} = \underline{\boldsymbol{0}} + \underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$ $\quad$ ($\underline{\boldsymbol{0}}$ is defined to be the vector for which each entry is the number 0.)

(iv) $\quad \underline{\boldsymbol{u}} + (-\underline{\boldsymbol{u}}) = (-\underline{\boldsymbol{u}}) + \underline{\boldsymbol{u}} = \underline{\boldsymbol{0}}$ ($-\underline{\boldsymbol{u}}$ is defined to be $-1 \cdot \underline{\boldsymbol{u}}$, i.e. the vector for which each entry is the opposite of the corresponding entry in $\underline{\boldsymbol{u}}$.)

(v) $\quad c(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = c\,\underline{\boldsymbol{u}} + c\,\underline{\boldsymbol{v}}$

(vi) $\quad (c + d)\underline{\boldsymbol{u}} = c\,\underline{\boldsymbol{u}} + d\,\underline{\boldsymbol{u}}$

(vii) $\quad c(d\,\underline{\boldsymbol{u}}) = (c\,d)\,\underline{\boldsymbol{u}}$

(viii) $\quad 1\,\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$.

Geometric interpretation of vectors

The space $\mathbb{R}^n$ may be thought of in two equivalent ways. In both cases, $\mathbb{R}^n$ consists of all possible $n - tuples$ of numbers:

(i) We can think of those $n - tuples$ as representing points, as we're used to doing for $n = 1, 2, 3$. In this case we can write

$$\mathbb{R}^n = \left\{ \left(x_1, x_2, \ldots, x_n \right), \ s.t.\ x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}.$$

(ii) We can think of those $n - tuples$ as representing vectors that we can add and scalar multiply. In this case we can write

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_n \end{bmatrix}, \ s.t.\ x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}.$$

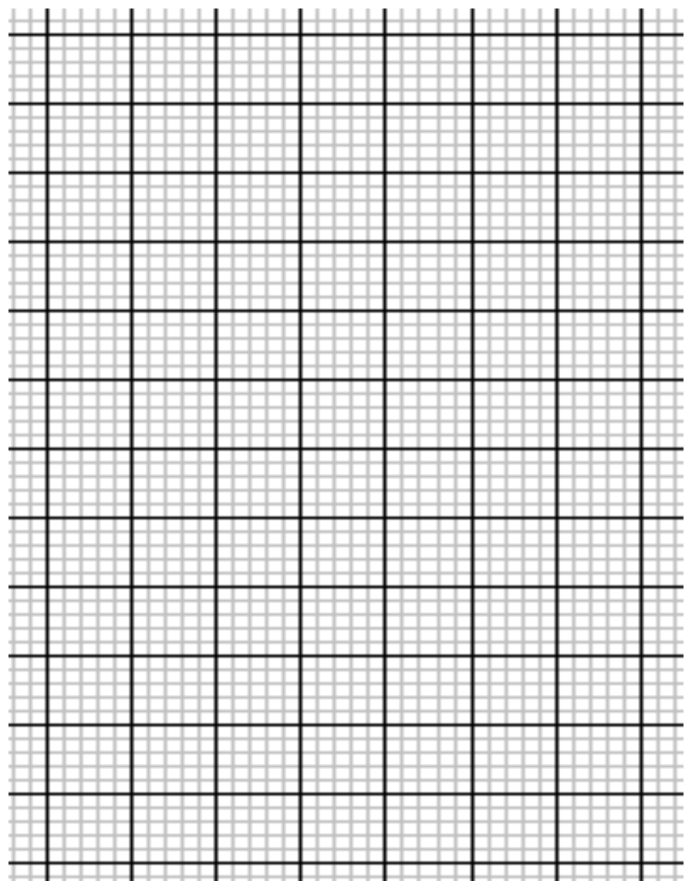
Since algebraic vectors (as above) can be used to measure geometric displacement, one can identify the two models of $\mathbb{R}^n$ as sets by identifying each point $\left(x_1, x_2, \ldots x_n \right)$ in the first model with the displacement vector $\underline{\boldsymbol{x}} = \left[x_1, x_2, \ldots x_m \right]^T$ from the origin to that point, in the second model, i.e. the "position vector" of the point.

Exercise 1) Let $\underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

1a) Plot the points $(1, -1)$ and $(1, 3)$, which have position vectors $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$. Draw these position vectors as arrows beginning at the origin and ending at the corresponding points.

1b) Compute $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}$ and then plot the point for which this is the position vector. Note that the algebraic operation of vector addition corresponds to the geometric process of composing horizontal and vertical displacements.

1c) Compute $3\,\underline{\boldsymbol{u}}$ and $-2\,\underline{\boldsymbol{v}}$ and plot the corresponding points for which these are the position vectors.



One of the key themes of this course is the idea of "linear combinations". These have an algebraic definition, as well as a geometric interpretation as combinations of displacements.

Definition: If we have a collection of n vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$ in $\mathbb{R}^n$, then any vector $\underline{\boldsymbol{v}} \in \mathbb{R}^n$ that can be expressed as a sum of scalar multiples of these vectors is called a *linear combination* of them. In other words, if we can write

$$\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_p\underline{\boldsymbol{v}}_p ,$$

then $\underline{\boldsymbol{v}}$ is a *linear combination* of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the *linear combination coefficients* or *weights*.

Example You've probably seen linear combinations in previous math/physics classes, even if you didn't realize it. For example you might have expressed the position vector $\underline{\mathbf{r}}$ as a linear combination

$$\underline{\mathbf{r}} = x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\boldsymbol{k}}$$

where $\underline{\mathbf{i}}, \underline{\mathbf{j}}, \underline{\mathbf{k}}$ represent the unit displacements in the x, y, z directions. Since we can express these displacements using Math 2270 notation as

$$\underline{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \underline{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we have

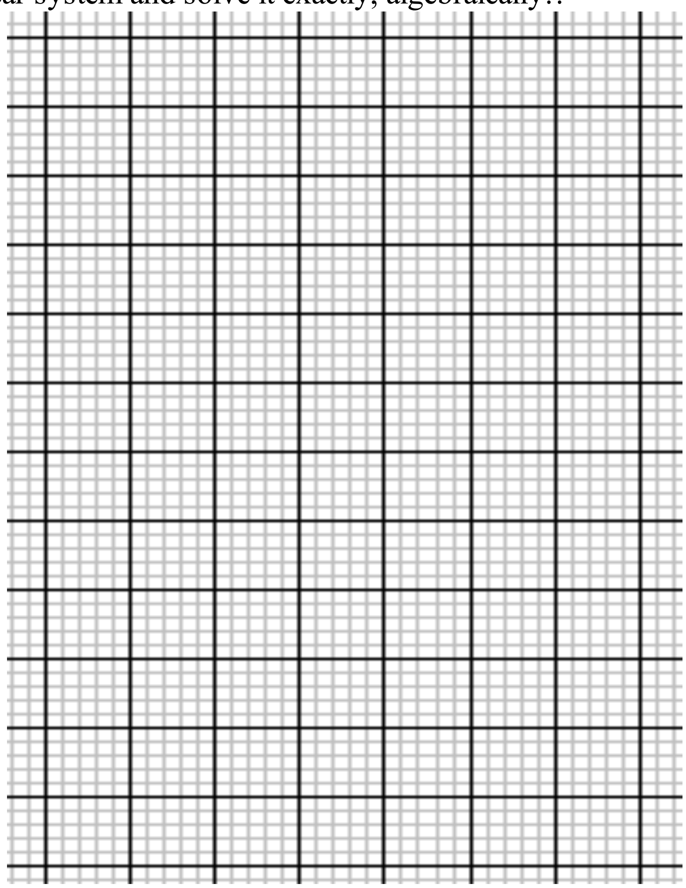
$$x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\boldsymbol{k}} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Exercise 2) Can you get to the point $(-2, 8) \in \mathbb{R}^2$, from the origin $(0, 0)$, by moving only in the $(\pm)$ directions of $\underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$? Algebraically, this means we want to solve the linear combination problem

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

2a) Superimpose a grid related to the displacement vectors $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$ onto the graph paper below, and, recalling that vector addition yields net displacement, and scalar multiplication yields scaled displacement, try to approximately solve the linear combination problem above, geometrically.

2b) Rewrite the linear combination problem as a linear system and solve it exactly, algebraically!!



2c) Can you get to any point (x, y) in $\mathbb{R}^2$, starting at $(0, 0)$ and moving only in directions parallel to $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$?

Argue geometrically and algebraically. How many ways are there to express $\begin{bmatrix} x \\ y \end{bmatrix}$ as a linear combination of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$?

Definition The *span* of a collection of vectors, written as $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$, is the collection of all linear combinations of those vectors.

Examples: We showed in 2c that $span\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}\} = \mathbb{R}^2$. On the other hand, $span\{\underline{\boldsymbol{u}}\}$ is the line with implicit equation $y = -x$.

Remark: The mathematical meaning of the word *span* is related to the English meaning - as in "wing span" or "span of a bridge", but it's also different. The span of a collection of vectors goes on and on and does not "stop" at the vector or associated endpoint:

What we may have realized in the previous exercise is the very important:

Fundamental Fact A *vector equation* (linear combination problem)

$$x_1\,\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \ ... \ + x_n\,\underline{\boldsymbol{a}}_n = \underline{\boldsymbol{b}}$$

is actually a system of linear equations for the unknown weights $x_1, x_2, ... x_n$; in fact the system of linear equations has augmented matrix given by

$$\left[\underline{\boldsymbol{a}}_1 \ \ \underline{\boldsymbol{a}}_2 \ \ \cdots \ \ \underline{\boldsymbol{a}}_n \ \ \underline{\boldsymbol{b}}\ \right]$$

(where we have expressed the augmented matrix in terms of its columns). In particular, $\underline{\boldsymbol{b}}$ can be generated by a linear combination of $\underline{\boldsymbol{a}}_1 \ \ \underline{\boldsymbol{a}}_2 \ \ ... \ \ \underline{\boldsymbol{a}}_n$ if and only if there exists a solution to the linear system corresponding to the augmented matrix above.

This fundamental fact is so important to the course, that we should check it in general at some point.

Exercise 3a) Does the vector equation

$$x_1\begin{bmatrix}1\\0\\2\end{bmatrix} + x_1\begin{bmatrix}-1\\2\\0\end{bmatrix} = \begin{bmatrix}2\\-3\\1\end{bmatrix}$$

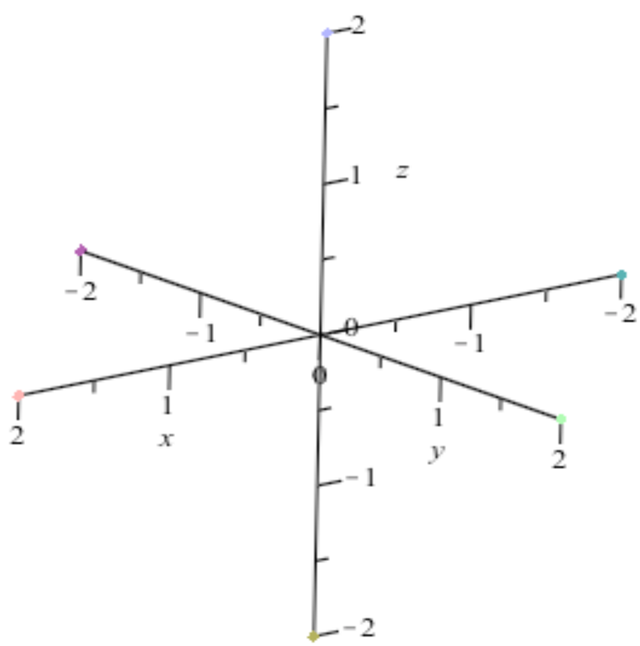
have any solutions?

3b) What geometric question is this related to? What geometric object is $span\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}-1\\2\\0\end{bmatrix}\right\}$?

3c) Use an augmented matrix calculation to find what condition needs to hold on vectors $\underline{\boldsymbol{b}}$ so that

$$\underline{\boldsymbol{b}} \in span\left\{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right\}. \quad (!!)$$

In case we want to sketch anything related to Exercise 3:



Wed Jan 17

- 1.4 the matrix equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$.

Announcements:

Warm-up Exercise:

Recall

Fundamental Fact   A *vector equation* (linear combination problem)

$$x_1\,\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \ldots + x_n\,\underline{\boldsymbol{a}}_n = \underline{\boldsymbol{b}}$$

is actually a system of linear equations for the unknown weights $x_1, x_2, \ldots.\, x_n$; in fact the system of linear equations has augmented matrix given by

$$\left[\underline{\boldsymbol{a}}_1\ \ \underline{\boldsymbol{a}}_2\ \cdots\ \underline{\boldsymbol{a}}_n\ \underline{\boldsymbol{b}}\ \right]$$

(where we have expressed the augmented matrix in terms of its columns). In particular, $\underline{\boldsymbol{b}}$ can be generated by a linear combination of $\underline{\boldsymbol{a}}_1\ \ \underline{\boldsymbol{a}}_2\ \ldots\ \underline{\boldsymbol{a}}_n$ if and only if there exists a solution to the linear system corresponding to the augmented matrix above.
We should check this carefully today, assuming we didn't do so on Tuesday:

Definition (from 1.4) If A is an $m \times n$ matrix, with columns $\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ \ldots\ \underline{\boldsymbol{a}}_n$ (in $\mathbb{R}^m$) and if $\underline{\boldsymbol{x}} \in \mathbb{R}^n$, then $A\,\underline{\boldsymbol{x}}$ is defined to be the linear combination of the columns, with weights given by the corresponding entries of $\underline{\boldsymbol{x}}$. In other words,

$$A\,\underline{\boldsymbol{x}} := x_1\,\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \ \ldots x_n\,\underline{\boldsymbol{a}}_n\ .$$

(This will give us a way to abbreviate vector equations.)

Definition. Let $\underline{u}$, $\underline{v}$ be vectors in $\mathbb{R}^n$. Then the *dot product* $\underline{u} \cdot \underline{v}$ is defined by

$$\underline{u} \cdot \underline{v} = \sum_{j=1}^{n} u_j v_j = u_1 v_1 + u_2 v_2 + \dots u_n v_n .$$

Computational Theorem: (This is usually a quicker way to compute $A\,\underline{x}$. Let If A be an $m \times n$ matrix, with rows $R_1, R_2, \dots R_m$. Then $A\,\underline{x}$ may also be computed using the rows of A and the dot product:

$$x_1 \underline{a}_1 + x_2 \underline{a}_2 + \dots x_n \underline{a}_n = A\,\underline{x} = \begin{bmatrix} R_1 \cdot \underline{x} \\ R_2 \cdot \underline{x} \\ : \\ R_m \cdot \underline{x} \end{bmatrix}$$

Exercise 1a) Compute both ways:

$$\begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} =$$

Exercise 1b) Write as a matrix times a vector:

$$3 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} =$$

Summary Theorem: (Three applications in one) If A is an $m \times n$ matrix, with columns $\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n$ (in $\mathbb{R}^m$) and if $\underline{\boldsymbol{b}} \in \mathbb{R}^m$, then the *matrix equation*

$$A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$$

has the same solution set as the *vector equation*

$$x_1\,\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \ \ldots x_n\,\underline{\boldsymbol{a}}_n = \underline{\boldsymbol{b}}$$

which, in turn, has the same solution set as the *system of linear equations* whose augmented matrix is

$$\left[\underline{\boldsymbol{a}}_1 \ \ \underline{\boldsymbol{a}}_2 \ \ \cdots \ \ \underline{\boldsymbol{a}}_n \ \ \underline{\boldsymbol{b}}\right].$$

Fri Jan 19

- 1.5 solution sets to matrix equations; homogeneous and non-homogeneous systems of equations.

Announcements:

Warm-up Exercise:

Definition: A system of linear equations is *homogeneous* if it can be written in the form

$$A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

where A is an $m \times n$ matrix, and $\underline{\mathbf{0}}$ is the zero vector in $\mathbb{R}^m$.

Definition: A system of linear equations is *nonhomogeneous* if it can be written in the form

$$A\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

where A is an $m \times n$ matrix, and $\underline{\mathbf{b}}$ is non-zero, i.e. not the zero vector in $\mathbb{R}^m$.

Our goal in section 1.5 is to understand the relationship between the solution sets of homogeneous and nonhomogeneous sytems, when the matrix A is the same.

To understand how the different solution sets are related, we will check and use these algebra facts:

$$A\,(\underline{\mathbf{x}} + \underline{\mathbf{y}}) = A\,\underline{\mathbf{x}} + A\,\underline{\mathbf{y}}$$

$$A\,(c\,\underline{\mathbf{x}}) = c\,A\,\underline{\mathbf{x}}\,.$$

Homogeneous systems: Notice that for any matrix A, it's always true that the homogeneous equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$ has a solution $\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$, so homogeneous systems are always consistent. The question is whether there are more solutions. (And, we call the solution $\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$ the "trivial" solution.)

Exercise 1) Find and compare the solution sets of the following two linear systems. The first one is homogeneous and the second one is non-homogeneous. How do the solutions sets appear to be related?

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned} \qquad\qquad \begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 7 \\ -3x_1 - 2x_2 + 4x_3 &= -1 \\ 6x_1 + x_2 - 8x_3 &= -4 \end{aligned}$$

What happened in Exercise 1 is what always happens when the non-homogeneous system is consistent. It says that for consistent nonhomogeneous systems, all solution sets are "translations" of each other.

Theorem (Fundamental Theorem of matrix equations) Suppose the equation $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ is consistent for some $\underline{\boldsymbol{b}}$. Let $\underline{\boldsymbol{p}}$ be a solution. Then the solution set of $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ is the set of all vectors

$$\underline{\boldsymbol{w}} = \underline{\boldsymbol{p}} + \underline{\boldsymbol{v}}_h$$

where $\underline{\boldsymbol{v}}_h$ is any solution of the homogeneous equation

$$A\,\underline{\mathbf{x}} = \underline{\boldsymbol{0}}.$$

We can verify why this theorem is true!

Room for more examples...