Syllabus for Math 2270-004 Differential Equations and Linear Algebra

Spring 2018

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office LCB 204, 801.581.7318
office hours M 2:00-3:00 p.m. LCB 204, T 4:30-6:00 location ~~TBA~~ JWB 240, and by appointment. Also available after class (briefly). → (may change)
Lecture MTWF 12:55-1:45 p.m. MWF in WEB L110, T in WEB L102 don't forget

Course websites

Daily lecture notes and weekly homework assignments will be posted on our public home page.
http://www.math.utah.edu/∼korevaar/2270spring18 up by tonight, hopefully

There are blank spaces in the notes where we will work out examples and fill in details together. Research has shown that class attendance with active participation - including individual and collaborative problem solving, and writing notes by hand - are effective ways to learn class material for almost everyone. Passively watching a lecture is not usually effective. Class notes will be posted at least several days before we use them, so that you have ample time to print them out. Printing for math classes is free in the Math Department Rushing Student Center, in the basement of LCB. Beyond what's outlined in the notes, there will often be additional class discussion related to homework and other problems.

Grades and exam material will be posted on our CANVAS course page; access via Campus Information Systems.

Textbook *Linear Algebra and its Applications*, 5th edition, by David D. Lay. ISBN: 032198238X

Final Exam logistics: Monday, April 30, 1:00-3:00 p.m., in our MWF classroom WEB L110. This is the University scheduled time and location.

Catalog description for Math 2270: Euclidean space, linear systems, Gaussian elimination, determinants, inverses, vector spaces, linear transformations, quadratic forms, least squares and linear programming, eigenvalues and eigenvectors, diagonalization. Includes theoretical and computer lab components.

Course Overview: This course is the first course in a year long sequence (2270-2280) devoted to linear mathematics. In this course, we study two objects: vectors and matrices. We start by thinking of vectors and matrices as arrays of numbers, then we progress to thinking of vectors as elements of a vector space and matrices as linear transformations. In our study of vectors and matrices, we learn to solve systems of linear equations, familiarize ourselves with matrix algebra, and explore the theory of vector spaces. Some key concepts we study are determinants, eigenvalues and eigenvectors, orthogonality, symmetric matrices, and quadratic forms. Along the way we encounter applications of the material in science and engineering. Students who continue into Math 2280 will see that linear algebra is one of the foundations, together with Calculus, upon which the study of differential equations is based.

Prerequisites: C or better in MATH 2210 or MATH 1260 or MATH 1321 or MATH 1320. Practically speaking, you are better prepared for this course if your grades in the prerequisite courses were above the "C" level.

Students with disabilities: The University of Utah seeks to provide equal access to its programs, services and activities for people with disabilities. If you will need accommodations in the class, reasonable prior notice needs to be given to the Center for Disability Services, 162 Olpin Union Building, 581-5020. CDS will work with you and the instructor to make arrangements for accommodations. All information in this course can be made available in alternative format with prior notification to the Center for Disability Services.

Grading

Math 2270-004 is graded on a curve. By this I mean that the final grading scale may end up lower than the usual 90/80/70% cut-offs. **note:** In order to receive a grade of at least "C" in the course you must earn a grade of at least "C" on the final exam. Typical grade distributions in Math 2270 end up with grades divided roughly in thirds between A's, B's, and the remaining grades. Individual classes vary.

Details about the content of each assignment type, and how much they count towards your final grade are as follows:

- Homework (15%): There will be one homework assignment each week. Homework problems will be posted on our public page, and homework assignments will be due in class on Wednesdays. Homework assignments must be stapled. Unstapled assignments will not receive credit. I understand that sometimes homework cannot be completed on time due to circumstances beyond your control. To account for this, each student will be allowed to turn in **three** late homework assignments throughout the course of the semester. These assignments cannot be turned in more than one week late, and must be turned in on a Wednesday with the next homework assignment. You do not need to tell me the reason why your homework assignment is late.

- Quizzes (10%): At the end of most Wednesday classes, a short 1-2 problem quiz will be given, taking roughly 10 minutes to do. The quiz will cover relevant topics from the week's lectures, homework, and food for thought work. Two of a student's lowest quiz scores will be dropped. There are no makeup quizzes. You will be allowed and encouraged to work together on these quizzes.

- Midterm exams (30%): Two class-length midterm exams will be given, On Friday February 9 ~~16~~ and Friday March ~~30~~ 16. I will schedule a room for review on the Thursday before each midterm, at our regular class time of 12:55-1:45 p.m. No midterm scores are dropped.

- Final exam (30%): A two-hour comprehensive exam will be given at the end of the semester. As with the midterms, a practice final will be posted. Please check the final exam time, which is the official University scheduled time. It is your responsibility to make yourself available for that time, so make any arrangements (e.g., with your employer) as early as possible.

- Food for Thought questions (15%): On Fridays without exams, we will spend half of the class time working on a collection of thought provoking problems. These problems will lend themselves naturally to discussion, and students will work with their learning communities to discuss/debate/ponder these questions. Students will turn in their Friday Food for Thought responses the following Monday in class, and these will be graded for completion. Solutions to Friday Food for Thoughts will be posted on Canvas. Your lowest Friday Food for Thought score will be dropped. (In other words, you can miss one Friday Food for Thought without penalty.)

Strategies for success

- Attend class regularly, and participate actively.
- Read or skim the relevant text book sections and lecture note outlines *before* you attend class.
- Ask questions and become involved.
- Plan to do homework daily; try homework on the same day that the material is covered in lecture; do not wait until just before homework and lab reports are due to begin serious work.
- Form study groups with other students.

Learning Objectives for 2270

Computation vs. Theory: This course is a combination of computational mathematics and theoretical mathematics. By theoretical mathematics, I mean abstract definitions and theorems, instead of calculations. The computational aspects of the course may feel more familiar and easier to grasp, but I urge you to focus on the theoretical aspects of the subject. Linear algebra is a tool that is heavily used in mathematics, engineering, and science, so it will likely be relevant to you later in your career. When this time comes, you will find that the computations of linear algebra can easily be done by computing systems such as Matlab, Maple, Mathematica or Wolfram alpha. But to understand the significance of these computations, a person must understand the theory of linear algebra. Understanding abstract mathematics is something that comes with practice, and takes more time than repeating a calculation. When you encounter an abstract concept in lecture, I encourage you to pause and give yourself some time to think about it. Try to give examples of the concept, and think about what the concept is good for.

The essential topics

Be able to find the solution set to linear systems of equations systematically, using row reduction techniques and reduced row echelon form - by hand for smaller systems and using technology for larger ones. Be able to solve (linear combination) vector equations using the same methods, as both concepts are united by the common matrix equation $A\mathbf{x} = \mathbf{b}$.

Be able to use the correspondence between matrices and linear transformations - first for transformations between $\mathbb{R}^n$ and $\mathbb{R}^m$, and later for transformations between arbitrary vector spaces.

Become fluent in matrix algebra techniques built out of matrix addition and multiplication, in order to solve matrix equations.

Understand the algebra and geometry of determinants so that you can compute determinants, with applications to matrix inverses and to oriented volume expansion factors for linear transformations.

Become fluent in the language and concepts related to general vector spaces: linear independence, span, basis, dimension, and rank for linear transformations. Understand how change of basis in the domain and range effect the matrix of a linear transformation.

Be able to find eigenvalues and eigenvectors for square matrices. Apply these matrix algebra concepts and matrix diagonalization to understand the geometry of linear transformations and certain discrete dynamical systems.

Understand how orthogonality and angles in $\mathbf{R}^2, \mathbf{R}^3$ generalize via the dot product to $\mathbf{R}^n$, and via general inner products to other vector spaces. Be able to use orthogonal projections and the Gram-Schmidt process, with applications to least squares problems and to function vector spaces.

Know the spectral theorem for symmetric matrices and be able to find their diagonalizations. Relate this to quadratic forms, constrained optimization problems, and to the singular value decomposition for matrices. Learn some applications to image processing and/or statistics.

Week-by-Week Topics Plan

Topic schedule is subject to slight modifications as the course progresses, but exam dates are fixed.

Week 1: 1.1-1.3; systems of linear equations, row reduction and echelon forms, vector equations.

Week 2: 1.4-1.6; matrix equations, solution sets of linear systems, applications.

Week 3: 1.7-1.9; linear dependence and independence, linear transformations and matrices.

Week 4: 2.1-2.3; matrix algebra and inverses, invertible matrices.

Week 5: 2.4-2.5; partitioned matrices and matrix factorizations. **Midterm exam 1 on Friday February 9** covering material from weeks 1-5.

Week 6: 3.1-3.3; determinants, algebraic and geometric properties and interpretations.

Week 7: 4.1-4.2, 2.8; vector spaces and subspaces, nullspaces and column spaces, general linear transformations.

Week 8: 4.3-4.6; linearly independent sets, bases, coordinate systems, dimension and rank

Week 9: 4.7, 5.1-5.2; change of basis, eigenvectors and how to find them.

Week 10: 5.3-5.4; diagonalization, eigenvectors and linear transformations. **Midterm exam 2 on Friday March 16** covering material from weeks 6-10

Week 11: 5.5-5.6, 6.1-6.2; complex eigendata, discrete dynamical systems, introduction to inner products and orthogonality.

Week 12: 6.3-6.5; orthogonal projections, Gram-Schmidt process, least squares problems.

Week 13: 6.7-6.8, 7.1; general inner product spaces and applications, diagonalization of symmetric matrices

Week 14: 7.2-7.4; quadtratic forms, constrained optimization, singular value decomposition.

Week 15: 7.5; applications to image processing and statistics.

Week16: Final exam Monday April 30, 1:00 - 3:00 p.m. in classroom WEB L110. This is the University scheduled time.

• 20 minutes syllabus
• rest of class

Math 2270-004 Week 1 notes

We will not necessarily finish the material from a given day's notes on that day. Or on an amazing day we may get farther than I've predicted. We may also add or subtract some material as the week progresses, but these notes represent an outline of what we will cover. These week 1 notes are for sections 1.1-1.3.

Monday January 8:

- Course Introduction
- 1.1: Systems of linear equations

• Go over course information on syllabus and course homepage:

http://www.math.utah.edu/~korevaar/2270spring18

• Note that there is a quiz this Wednesday on the material we cover today and tomorrow. Your first homework assignment will be due next Wednesday, January 17.

Then, let's begin!

• What is a linear equation? What is a system of linear equations?

any equation in some number of variables $x_1, x_2, \ldots x_n$ which can be written in the form

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

where b and the coefficients $a_1, a_2, \ldots a_n$ are real or complex numbers usually known in advance.

Question: Why do you think we call an equation like that $\boxed{\text{"linear"}}$?

one reason: in 2 variables, the collection of all solutions is a line.
i.e. $ax + by = c$

another: variables appear in a "linear" way, i.e. as multiples of each variable, to the 1st power

Exercise 1) Which of these is a linear equation?

1a) For the variables x, y

$$3x + 4y = 6$$

yes

1b) For the variables s, t

yes

$$2t = 5 - \sqrt{3}\, s$$

1c) For the variables x, y

no

$$2x = 5 - 3\sqrt{y}$$

variables only appear with power 1

- Then the general linear system (LS) of m equations in the n variables $x_1, x_2, \ldots x_n$ can be written as

$$\text{LS}\left\{\begin{array}{l} a_{11}\,x_1 + a_{12}\,x_2 + \ldots + a_{1\,n}\,x_n = b_1 \\ a_{21}\,x_1 + a_{22}\,x_2 + \ldots + a_{2\,n}\,x_n = b_2 \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \quad \vdots \\ a_{ml}\,x_1 + a_{m2}\,x_2 + \ldots + a_{m\,n}\,x_n = b_m \end{array}\right.$$

2 : 2nd eqn
n : multiplying nth variables

In such a linear systems the coefficients $a_{i\,j}$ and the right-side number b_j are usually known. The goal is to find values for the vector $\underline{\boldsymbol{x}} = \left[x_1, x_2, \ldots x_n\right]$ of variables so that all equations are true. (Thus this is often called finding "simultaneous" solutions to the linear system, because all equations will be true at once.)

Definition *The solution set of a system of linear equations is the collection of all solution vectors to that system.*

Notice that we use two subscripts for the coefficients $a_{i\,j}$ and that the first one indicates which equation it appears in, and the second one indicates which variable its multiplying; in the corresponding *coefficient matrix A* , this numbering corresponds to the row and column of $a_{i\,j}$:

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1\,n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2\,n} \\ \vdots & & & & \vdots \\ a_{ml} & a_{m2} & a_{m3} & \ldots & a_{m\,n} \end{bmatrix}$$

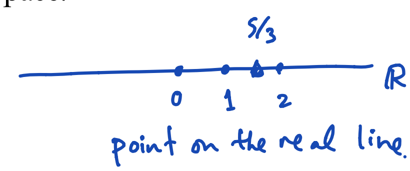
Let's start small, where geometric reasoning will help us understand what's going on when we look for solutions to linear equations and to linear systems of equations.

Exercise 2: Describe the solution set of each single linear equation below; describe and sketch its geometric realization in the indicated Euclidean space.

1 eqn, 1 unknown. the real numbers

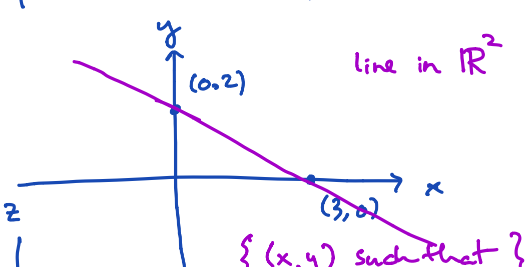
2a) $3\,x = 5$, for $x \in \mathbb{R}$.

$x = 5/3$ is an element of



2b) $2\,x + 3\,y = 6$, for $[x, y] \in \mathbb{R}^2$.

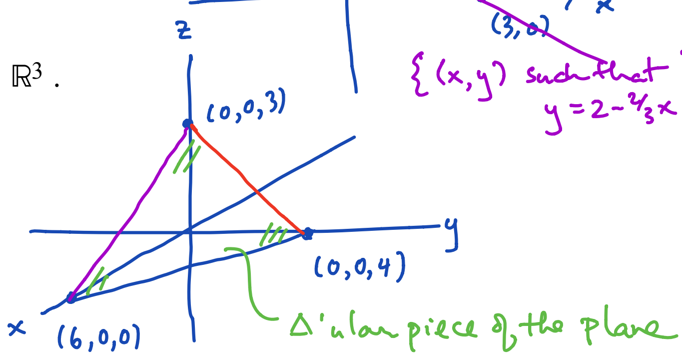
eg slope intercept form.
$3y = 6 - 2x$
$y = 2 - \frac{2}{3}x$
line in $\mathbb{R}^2$



2c) $2\,x + 3\,y + 4\mathbf{z} = 12$, for $[x, y, \mathbf{z}] \in \mathbb{R}^3$.

solution set is a plane

$z = 3 - \frac{1}{2}x - \frac{3}{4}y$



if $z = 0$, $2x + 3y = 12$
$y = 0$, $2x + 4z = 12$
$x = 0$, $3y + 4z = 12$

2 linear equations in 2 unknowns:

$$a_{11}\, x + a_{12}\, y = b_1$$
$$a_{21}\, x + a_{22}\, y = b_2$$

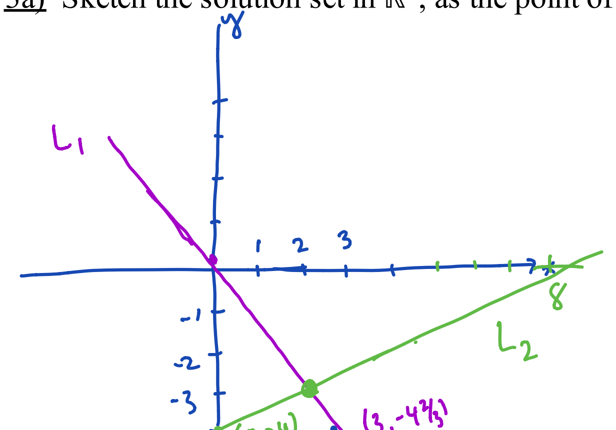
goal: find all $[x, y]$ making both of these equations true at once. Since the solution set to each single equation is a line one geometric interpretation is that you are looking for the intersection of two lines.

Exercise 3: Consider the system of two equations E_1, E_2:

$15 + 3y = 1$
$3y = -14$

$$E_1 \qquad 5\,x + 3\,y = 1 \leftarrow (0, \tfrac{1}{3}), \quad (3, -4\tfrac{2}{3})$$
$$E_2 \qquad x - 2\,y = 8 \quad (8, 0)$$

3a) Sketch the solution set in $\mathbb{R}^2$, as the point of intersection between two lines.



3b) Use the following three "elementary equation operations" to systematically reduce the system E_1, E_2 to an equivalent system (i.e. one that has the same solution set), but of the form

$$1\,x + 0\,y = c_1$$
$$0\,x + 1\,y = c_2$$

(so that the solution is $x = c_1, y = c_2$). Make sketches of the intersecting lines, at each stage.

The three types of elementary equation operation are below. Can you explain why the solution set to the modified system is the same as the solution set before you make the modification?

- interchange the order of the equations
- multiply one of the equations by a non-zero constant
- replace an equation with its sum with a multiple of a different equation.

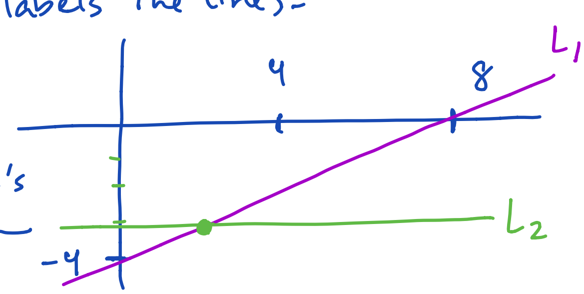
$\boxed{\text{HW} \quad 1.1 \ \# \ 1, 3, 13, 17}$

$E_2 \to E_1$ $\quad x - 2y = 8$ $\quad$ to get $1\cdot x$ in 1^{st} eq^n

$E_1 \to E_2$ $\quad 5x + 3y = 1$ $\quad$ just relabels the lines.

$x - 2y = 8$

$-5E_1 + E_2 \to E_2$ $\quad 13y = -39$ $\quad$ eliminate x's in 2^{nd} eq^n

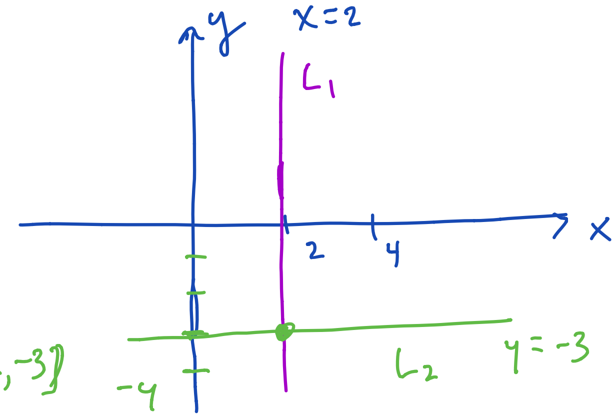


$$x - 2y = 8$$

$$\frac{E_2}{13} \to E_2 \qquad y = -3$$

lines are the same. We do this step to make it easy to eliminate the y's in eqtn 1

$$2E_2 + E_1 \to E_1 \qquad x = 2$$
$$y = -3$$



sol'n:
$[x, y] = [2, -3]$

Since elementary eqtn operations don't change the solution set to a system of linear eqtns, $[x, y] = [2, -3]$ is the sol'n to original system:

$x - 2y = 8$: $2 + 6 = 8$ ✓
$5x + 3y = 1$: $10 - 9 = 1$ ✓

3c) Look at your work in 3b. Notice that you could have save a lot of writing by doing this computation "synthetically", i.e. by just keeping track of the coefficients and right-side values. Using R_1, R_2 as symbols for the rows, your work might look like the computation below. Notice that when you operate synthetically the "elementary equation operations" correspond to "elementary row operations":

- interchange two rows
- multiply a row by a non-zero number
- replace a row by its sum with a multiple of another row.

more complete notation:

$$\left[\begin{array}{cc|c} 5 & 3 & 1 \\ 1 & -2 & 8 \end{array}\right]$$

$R_2 \to R_1$
$R_1 \to R_2$

$$\begin{array}{c} R_2 \\ R_1 \end{array}\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 5 & 3 & 1 \end{array}\right]$$

$-5R_1+R_2 \to R_2$

$$\begin{array}{c} \\ -5R_1+R_2 \end{array}\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 13 & -39 \end{array}\right]$$

$\frac{R_2}{13} \to R_2$

$$\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 1 & -3 \end{array}\right]$$

$2R_2+R_1 \to R_1$

$$\begin{array}{c} 2R_2+R_1 \\ \\ \end{array}\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right] \longrightarrow \begin{array}{l} x = 2 \\ y = -3 \end{array} \;!$$

3d) What are the possible geometric solution sets to 1, 2, 3, 4 or any number of linear equations in two unknowns?

empty set	(distinct parallel lines)
single point	(all lines have a common point of intersection)
line	(all lines are actually the same line)
all of $\mathbb{R}^2$	(all eqtns are $0x + 0y = 0$)

Math 2270-004
Tues Jan 9

- 1.1 Systems of linear equations
- 1.2 Row reduction and echelon form

Announcements:

- finish Monday's notes
- quiz tomorrow, §1.1, 1.2

Warm-up Exercise: was something like:

a) Do the lines

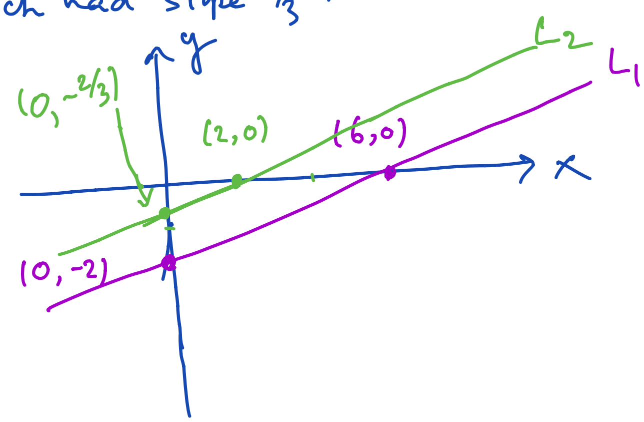
$E_1 \quad x - 3y = 6$

$E_2 \quad -2x + 6y = -4$

intersect?

b) Explain.

these lines were parallel
(the each had slope $\frac{1}{3}$)



$$\left[\begin{array}{cc|c} 1 & -3 & 6 \\ -2 & 6 & -4 \end{array}\right]$$

$$\begin{array}{l} \\ 2R_1 + R_2 \to R_2 \end{array} \left[\begin{array}{cc|c} 1 & -3 & 6 \\ 0 & 0 & 8 \end{array}\right] \leftarrow \text{says}$$

$0x + 0y = 8$

no solutions!

so the original system has no solutions

"inconsistent"

lines don't intersect!

Continuing our discussion of solution sets to systems of equations from yesterday:

<u>Solutions to linear equations in 3 unknowns:</u>

What is the geometric question we're answering in these cases?

<u>Exercise 1)</u> Consider the system

$$x + 2y + z = 4$$
$$3x + 8y + 7z = 20$$
$$2x + 7y + 9z = 23\,.$$

Use elementary equation operations (or if you prefer, elementary row operations in the synthetic version) to find the solution set to this system. There's a systematic way to do this, which we'll talk about today. It's called <u>Gaussian elimination.</u> (Check Wikipedia about Gauss - he was amazing!)

<u>Hint:</u> The solution set is a single point, $[x, y, z] = [5, -2, 3]$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right]$$

$$\begin{array}{l} \\ -3R_1+R_2\to R_2 \\ -2R_1+R_3\to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{array}\right]$$

$$\begin{array}{l} \\ R_2/2\to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 7 & 15 \end{array}\right]$$

$$\begin{array}{l} \\ \\ -3R_2+R_3\to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{l} -R_3+R_1\to R_1 \\ -2R_3+R_2\to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{l} -2R_2+R_1\to R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \longrightarrow \begin{array}{l} x = 5 \\ y = -2 \\ z = 3 \end{array}$$

Exercise 2) There are other possibilities. In the two systems below we kept all of the coeffients the same as in Exercise 1, except for a_{33}, and we changed the right side in the third equation, for 2a. Work out what happens in each case.

2a)

$$\begin{aligned} x + 2y + z &= 4 \\ 3x + 8y + 7z &= 20 \\ 2x + 7y + 8z &= 20\,. \end{aligned}$$

2b)

$$\begin{aligned} x + 2y + z &= 4 \\ 3x + 8y + 7z &= 20 \\ 2x + 7y + 8z &= 23\,. \end{aligned}$$

2c) What are the possible solution sets (and geometric configurations) for 1, 2, 3, 4,... equations in 3 unknowns?

$$\left[\begin{array}{ccc|c|c} 1 & 2 & 1 & 4 & 4 \\ 3 & 8 & 7 & 20 & 20 \\ 2 & 7 & 8 & 20 & 23 \end{array}\right]$$

$-3R_1+R_2 \to R_2$, $-2R_1+R_3 \to R_3$

$$\left[\begin{array}{ccc|cc} 1 & 2 & 1 & 4 & 4 \\ 0 & 2 & 4 & 8 & 8 \\ 0 & 3 & 6 & 12 & 15 \end{array}\right]$$

$R_2/2 \to R_2$

$$\left[\begin{array}{ccc|cc} 1 & 2 & 1 & 4 & 4 \\ 0 & 1 & 2 & 4 & 4 \\ 0 & 3 & 6 & 12 & 15 \end{array}\right]$$

$-3R_2+R_3 \to R_3$

$$\left[\begin{array}{ccc|cc} 1 & 2 & 1 & 4 & 4 \\ 0 & 1 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right]$$

$R_3/3 \to R_3$

$$\left[\begin{array}{ccc|c|c} 1 & 2 & 1 & 4 & 4 \\ 0 & 1 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

row echelon form.

for 2b). 3rd eqtn says $0x + 0y + 0z = 1$
so system 2b has no soltns
"inconsistent system."

$-2R_2+R_1 \to R_1$

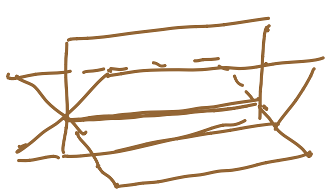
$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

reduced row echelon form.

for 2a):
$$\begin{aligned} x \quad -3z &= -4 \\ y + 2z &= 4 \end{aligned}$$

solution set
$$\begin{aligned} x &= -4+3z \\ y &= 4-2z \\ z &= \text{free} \\ &(z \in \mathbb{R}) \end{aligned}$$

a line in $\mathbb{R}^3$



After Wed, all §1.1 HW is accessible
all §1.2 HW is accessible. (21, 25, 27, 31 will make you think)

Summary of the systematic method known as Gaussian elimination for solving systems of linear equations.

We write the linear system (LS) of m equations for the vector $\underline{\boldsymbol{x}} = [x_1, x_2, \ldots x_n]$ of the n unknowns as

$$\begin{aligned} a_{11}\,x_1 + a_{12}\,x_2 + \ldots + a_{1\,n}\,x_n &= b_1 \\ a_{21}\,x_1 + a_{22}\,x_2 + \ldots + a_{2\,n}\,x_n &= b_2 \\ \vdots \qquad\qquad\qquad & \vdots \quad \vdots \\ a_{m1}\,x_1 + a_{m2}\,x_2 + \ldots + a_{m\,n}\,x_n &= b_m \end{aligned}$$

The matrix that we get by adjoining (augmenting) the right-side $\underline{\boldsymbol{b}}$-vector to the coefficient matrix $A = [a_{i\,j}]$ is called the *augmented matrix* $\langle A|\underline{\boldsymbol{b}}\rangle$:

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1\,n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2\,n} & b_2 \\ \vdots & & & & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{m\,n} & b_m \end{array}\right]$$

Our goal is to find all the solution vectors $\underline{\boldsymbol{x}}$ to the system - i.e. the *solution set* .

There are three types of *elementary equation operations* that don't change the solution set to the linear system. They are

- interchange two of equations
- multiply one of the equations by a non-zero constant
- replace an equation with its sum with a multiple of a different equation.

And that when working with the augmented matrix $\langle A|\underline{\boldsymbol{b}}\rangle$ these correspond to the three types of *elementary row operations:*

- interchange ("swap") two rows
- multiply one of the rows by a non-zero constant
- replace a row by its sum with a multiple of a different row.

Gaussian elimination: Use elementary row operations and work column by column (from left to right) and row by row (from top to bottom) to first get the augmented matrix for an equivalent system of equations which is in

row-echelon form:

(1) All "zero" rows (having all entries $= 0$) lie beneath the non-zero rows.

(2) The leading (first) non-zero entry in each non-zero row lies strictly to the right of the one above it. These entries are called *pivots* in our text, and the corresponding columns are called *pivot columns*.

(At this stage you could "backsolve" to find all solutions.)

Next, continue but by working from bottom to top and from right to left instead, so that you end with an augmented matrix that is in

reduced row echelon form: (1),(2), together with

(3) Each leading non-zero row entry has value 1 . Such entries are called "leading 1'*s* " and are the

"pivots" for the corresponding pivot columns.

(4) Each pivot column has $0's$ in all the entries except for the pivot entry of 1.

Finally, read off how to explicitly specify the solution set, by "backsolving" from the reduced row echelon form.

Note: There are lots of row-echelon forms for a matrix, but only one reduced row-echelon form. (We'll see why later.) All mathematical software packages have a command to find the reduced row echelon form of a matrix.

<u>Exercise 3</u> Find all solutions to the system of 3 linear equations in 5 unknowns

$$\begin{aligned} x_1 - 2\,x_2 + 3\,x_3 + 2\,x_4 + x_5 &= 10 \\ 2\,x_1 - 4\,x_2 + 8\,x_3 + 3\,x_4 + 10\,x_5 &= 7 \\ 3\,x_1 - 6\,x_2 + 10\,x_3 + 6\,x_4 + 5\,x_5 &= 27\,. \end{aligned}$$

Here's the augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array}\right]$$

Find the reduced row echelon form of this augmented matrix and then backsolve to explicitly parameterize the solution set. (Hint: it's a two-dimensional plane in $\mathbb{R}^5$, if that helps. :-))

Maple says:

```
> with(LinearAlgebra) :   # matrix and linear algebra library
> A := Matrix(3, 5, [1, -2, 3, 2, 1,
                     2, -4, 8, 3, 10,
                     3, -6, 10, 6, 5]) :
  b := Vector([10, 7, 27]) :
  <A|b>;     # the mathematical augmented matrix doesn't actually have
               # a vertical line between the end of A and the start of b
  ReducedRowEchelonForm(<A|b>);
```

$$\begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{bmatrix} \tag{1}$$

```
> LinearSolve(A, b);
      #  this command will actually write down the general solution, using
      # Maple's way of writing free parameters, which actually makes
      # some sense.  Generally when there  are free parameters involved,
      #  there will be equivalent ways to express the solution that may
      # look different.  But usually Maple's version will look like yours,
      # because it's using the same algorithm and choosing the free parameters
      # the same way too.
```

$$\begin{bmatrix} 5 + 2\,\_t_2 - 3\,\_t_5 \\ \_t_2 \\ -3 - 2\,\_t_5 \\ 7 + 4\,\_t_5 \\ \_t_5 \end{bmatrix} \tag{2}$$

```
>
```

Math 2270-004
Wed Jan 10

- 1.3 Vectors and vector equations
- Quiz today at end of class, on section 1.1-1.2 material

Announcements:

- Finish Tues notes, start 1.3 on Friday
  also on Fri, "f.f.t" problems
- Quiz today.

Warm-up Exercise: until 12:58

Do these 3 lines in the plane intersect?
Use an augmented matrix & el. row ops to answer.

$3x - 5y = 1$
$x + 2y = 4$
$-x + 3y = 1$

$3\cdot 2 - 5\cdot 1 = 1$ ✓
$2 + 2\cdot 1 = 4$ ✓
$-2 + 3\cdot 1 = 1$ ✓

lines do intersect
at [2, 1].

$$\left[\begin{array}{cc|c} 3 & -5 & 1 \\ 1 & 2 & 4 \\ -1 & 3 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_1 \\ R_1 \to R_2 \\ \\ \end{array} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & -5 & 1 \\ -1 & 3 & 1 \end{array}\right]$$ pivot

$$\begin{array}{l} \\ -3R_1 + R_2 \to R_2 \\ R_1 + R_3 \end{array} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -11 & -11 \\ 0 & 5 & 5 \end{array}\right]$$

$$\begin{array}{l} \\ \frac{R_2}{-11} \to R_2 \\ \\ \end{array} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{array}\right]$$

$$\begin{array}{l} \\ \\ -5R_2 + R_3 \end{array} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{l} -2R_2 + R_1 \to R_1 \\ \\ \\ \end{array} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

"unclock" $1x + 0y = 2$
$0x + 1y = 1$

$$\boxed{\begin{array}{l} x = 2 \\ y = 1 \end{array}}$$

1.3 Vectors and vector equations: We'll carefully define vectors, algebraic operations on vectors and geometric interpretations of these operations, in terms of displacements. These ideas will eventually give us another way to interpret systems of linear equations.

<u>Definition</u>: A matrix with only one column, i.e an $n \times 1$ matrix, is a *vector* in $\mathbb{R}^n$. (We also call matrices with only one row "vectors", but in this section our vectors will always be column vectors.)

Examples:

2210 1320

$\underline{\boldsymbol{u}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is a vector in $\mathbb{R}^2$. $\langle 3, -1 \rangle$ (same)

$3\hat{i} - \hat{j}$

$\underline{\boldsymbol{v}} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 5 \\ 0 \end{bmatrix}$ is a vector in $\mathbb{R}^5$. ?

- We can multiply vectors by *scalars* (real numbers) and add together vectors that are the same size. We can combine these operations.

<u>Example:</u> For $\underline{\boldsymbol{u}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{w}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ compute

$$\underline{\boldsymbol{u}} + 4\underline{\boldsymbol{w}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \underbrace{4\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

<u>Definition:</u> For $\underline{\boldsymbol{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \underline{\boldsymbol{v}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n; c \in \mathbb{R},$

$$\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} := \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ : \\ u_n + v_n \end{bmatrix} \quad ; \quad c\,\underline{u} := \begin{bmatrix} cu_1 \\ cu_2 \\ : \\ cu_n \end{bmatrix}.$$

Exercise 1 Our vector notation may not be the same as what you used in math 2210 or Math 1320 or 1321. Let's discuss the notation you used, and how it corresponds to what we're doing here.

did

Vector addition and scalar multiplication have nice algebraic properties:

Let $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}} \in \mathbb{R}^n$, $c, d \in \mathbb{R}$. Then

(i) $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} = \underline{\boldsymbol{v}} + \underline{\boldsymbol{u}}$

(ii) $(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) + \underline{\boldsymbol{w}} = \underline{\boldsymbol{u}} + (\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}})$

(iii) $\underline{\boldsymbol{u}} + \underline{\mathbf{0}} = \underline{\mathbf{0}} + \underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$

(iv) $\underline{\boldsymbol{u}} + (-\underline{\boldsymbol{u}}) = (-\underline{\boldsymbol{u}}) + \underline{\boldsymbol{u}} = \underline{\mathbf{0}}$

(v) $c(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = c\,\underline{u} + c\,\underline{\boldsymbol{v}}$

(vi) $(c + d)\underline{\boldsymbol{u}} = c\,\underline{\boldsymbol{u}} + d\,\underline{\boldsymbol{u}}$

(vii) $c(d\,\underline{\boldsymbol{u}}) = (c\,d)\,\underline{\boldsymbol{u}}$

(viii) $1\,\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$.

<u>Exercise 2</u>. Verify why these properties hold!

Geometric interpretation of vectors

The space $\mathbb{R}^n$ may be thought of in two equivalent ways. In both cases, $\mathbb{R}^n$ consists of all possible $n - tuples$ of numbers:

(i) We can think of those $n - tuples$ as representing points, as we're used to doing for $n = 1, 2, 3$. In this case we can write

$$\mathbb{R}^n = \left\{ \left(x_1, x_2, ..., x_n\right), s.t.\ x_1, x_2, ..., x_n \in \mathbb{R} \right\}.$$

(ii) We can think of those $n - tuples$ as representing vectors that we can add and scalar multiply. In this case we can write

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_n \end{bmatrix}, \ s.t.\ x_1, x_2, ..., x_n \in \mathbb{R} \right\}.$$

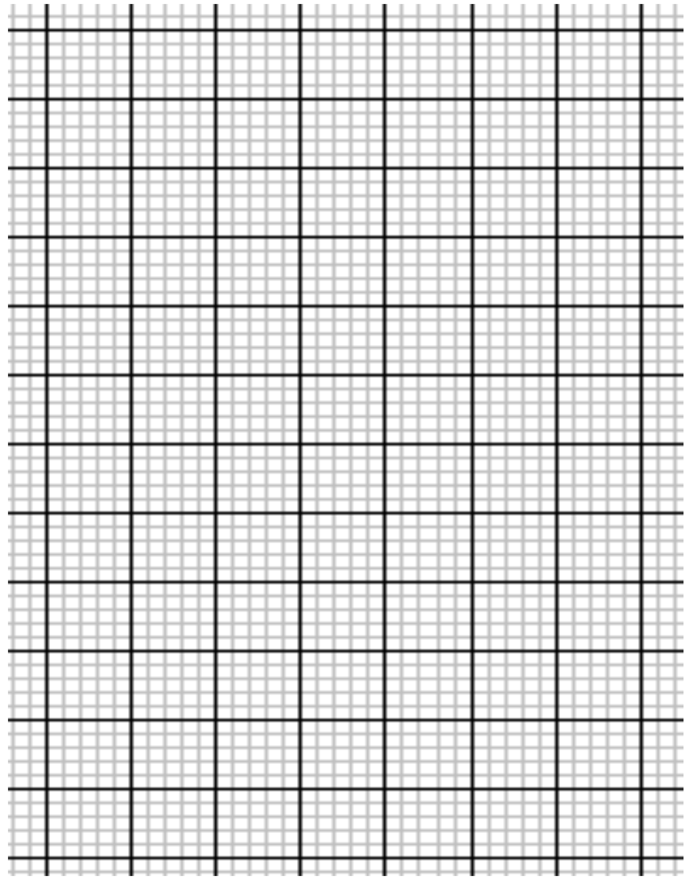
Since algebraic vectors (as above) can be used to measure geometric displacement, one can identify the two models of $\mathbb{R}^n$ as sets by identifying each point $\left(x_1, x_2, ...x_n\right)$ in the first model with the displacement vector $\underline{\boldsymbol{x}} = \left[x_1, x_2, ...x_m\right]^T$ from the origin to that point, in the second model, i.e. the position vector.

Exercise 3) Let $\underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

1a) Plot the points $(1, -1)$ and $(1, 3)$, which have position vectors $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$. Draw these position vectors as arrows beginning at the origin and ending at the corresponding points.

1b) Compute $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}$ and then plot the point for which this is the position vector. Note that the algebraic operation of vector addition corresponds to the geometric process of composing horizontal and vertical displacements.

1c) Compute $3\,\underline{\boldsymbol{u}}$ and $-2\,\underline{\boldsymbol{v}}$ and plot the corresponding points for which these are the position vectors.



One of the key themes of this course is the idea of "linear combinations". These have an algebraic definition, as well as a geometric interpretation as combinations of displacements.

Definition: If we have a collection of n vectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_p\}$ in $\mathbb{R}^n$, then any vector $\underline{v} \in \mathbb{R}^n$ that can be expressed as a sum of scalar multiples of these vectors is called a *linear combination* of them. In other words, if we can write

$$\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_p\underline{v}_p \,,$$

then $\underline{v}$ is a *linear combination* of $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the *linear combination coefficients* or *weights*.

Example You've probably seen linear combinations in previous math/physics classes, even if you didn't realize it. For example you might have expressed the position vector $\underline{\mathbf{r}}$ as a linear combination

$$\underline{\mathbf{r}} = x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\mathbf{k}}$$

where $\underline{\mathbf{i}}, \underline{\mathbf{j}}, \underline{\mathbf{k}}$ represent the unit displacements in the x, y, z directions. Since we can express these displacements using Math 2270 notation as

$$\underline{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \underline{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we have

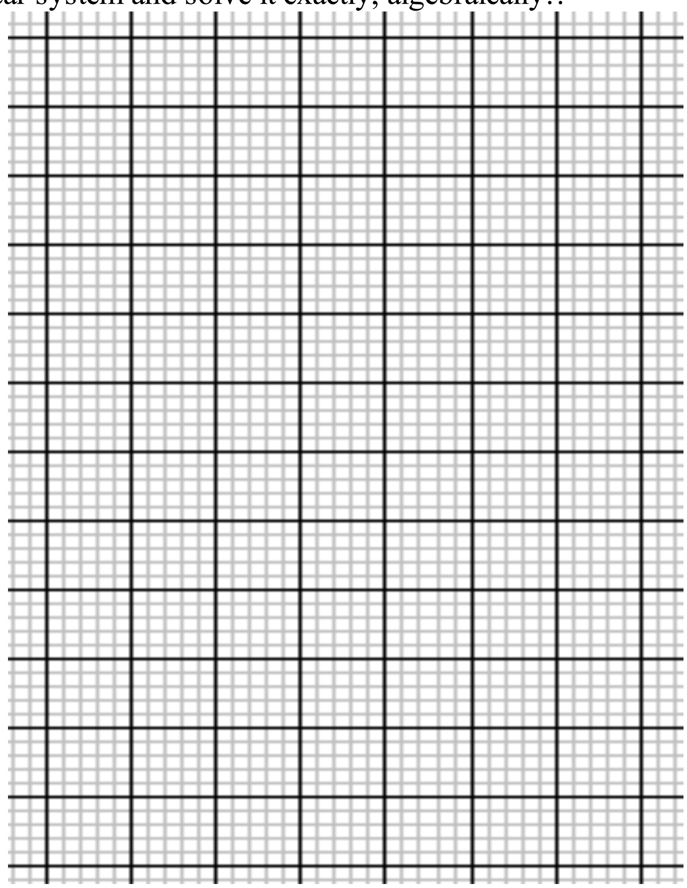
$$x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\mathbf{k}} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Exercise 4) Can you get to the point $(-2, 8) \in \mathbb{R}^2$, from the origin $(0, 0)$, by moving only in the $(\pm)$ directions of $\underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$? Algebraically, this means we want to solve the linear combination problem

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

4a) Superimpose a grid related to the displacement vectors $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}$ onto the graph paper below, and, recalling that vector addition yields net displacement, and scalar multiplication yields scaled displacement, try to approximately solve the linear combination problem above, geometrically.

4b) Rewrite the linear combination problem as a linear system and solve it exactly, algebraically!!



1c) Can you get to any point (x, y) in $\mathbb{R}^2$, starting at $(0, 0)$ and moving only in directions parallel to $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}$?

Argue geometrically and algebraically. How many ways are there to express $\begin{bmatrix} x \\ y \end{bmatrix}$ as a linear combination of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$?

Math 2250-004
Fri Jan 12

- 1.3 linear combinations and vector equations, continued.
- food for thought in second half of class

• try to finish 1.1 & 1.2 HW
(try 1.3 if you're bold)

Announcements:

• MLK day is Monday

• Use Wed notes today (before f.f.t. lab).
1.3 due Tues.

• I'll stay late today (until 2:45
• I'll bring class notes for next week on Tuesday

Warm-up Exercise: • Which of these augmented matrices is in reduced row echelon form?

• Do the associated systems of linear equations have
(i) no solutions (inconsistent)
(ii) a unique solution (i.e. exactly one sol'tn)
(iii) ∞'ly many solutions

unique sol'tn a) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 15 \end{array}\right]$ $\neq 0$, $\neq 1$ NO

no sol'tns, (inconsistent) last eqtn says $0=1$ b) $\left[\begin{array}{ccc|c} 1 & 0 & 17 & \cancel{3} \\ 0 & 1 & 6 & \cancel{2} \\ 0 & 0 & 0 & 1 \end{array}\right]$ needed to be zero NO pivots are O.K. $\left[\begin{array}{ccc|c} 1 & 0 & 17 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ is in rref

unique $y=2$, $x=5$ c) $\left[\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right]$ NO 1st non-zero row entries don't move right as you go down rows

∞'ly many (1 free parameter because 3rd column is not pivot column d) $\left[\begin{array}{ccc|c} 1 & 0 & 17 & 3 \\ 0 & 1 & 16 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ YES!

$$x + 17z = 3$$
$$y + 16z = 2$$

Sol'tns are
$x = 3 - 17z$
$y = 2 - 16z$
z free

1.3 Linear combinations and linear systems of equations, continued.

Fundamental Fact A vector equation (linear combination problem)

$$x_1\underline{\boldsymbol{v}}_1 + x_2\underline{\boldsymbol{v}}_2 + \ ... \ + x_p\underline{\boldsymbol{v}}_p = \underline{\boldsymbol{b}}$$

has the same solution set as the linear system whose augmented matrix is given by

$$\left[\underline{\boldsymbol{v}}_1 \ \ \underline{\boldsymbol{v}}_2 \ \cdots \ \underline{\boldsymbol{v}}_p \ \underline{\boldsymbol{b}}\ \right]$$

In particular, $\underline{\boldsymbol{b}}$ can be generated by a linear combination of $\underline{\boldsymbol{v}}_1 \ \ \underline{\boldsymbol{v}}_2 \ ... \ \underline{\boldsymbol{v}}_p$ if and only if there exists a solution to the linear system corresponding to the augmented matrix above.

Definition: The span of a collection of vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ ... \ \underline{\boldsymbol{v}}_p$ in $\mathbb{R}^n$ is the collection of all vectors $\underline{\boldsymbol{w}}$ which can be expressed as linear combinations of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ ... \ \underline{\boldsymbol{v}}_p$. We denote this collection as

$$span\{\ \underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ ... \ \underline{\boldsymbol{v}}_p\}.$$

Remark: The mathematical meaning of the word span is related to the English meaning - as in "wing span" or "span of a bridge", but it's also different. The span of a collection of vectors goes on and on and does not "stop" at the vector or associated endpoint:

Example 1)

- In Exercise 4 yesterday, consider the $span\{\underline{\boldsymbol{u}}\} = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$. This is the set of all vectors of the

form $\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$ with free parameter $x_1 \in \mathbb{R}$. This is a line through the origin of $\mathbb{R}^2$ described parametrically, that we're more used to describing with implicit equation $y = -x$ (which is short for $\{(x, y) \in \mathbb{R}^2 \ s.t.\ y = -x\}$). (More precisely, $span\{\underline{\boldsymbol{u}}\}$ is the collection of all position vectors for that line.)

Example 2:

- In Exercise 1 we showed that the span of $\underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is all of $\mathbb{R}^2$.

Exercise 1) Consider the two vectors $\underline{\mathbf{v}}_1 = [1, 0, 2]^T$, $\underline{\mathbf{v}}_2 = [-1, 2, 0]^T \in \mathbb{R}^3$.
1a) Sketch these two vectors as position vectors in $\mathbb{R}^3$, using the axes below.

1b) What geometric object is $span\{\underline{\mathbf{v}}_1\}$? Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.

1c) What geometric object is $span\{\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2\}$? Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.