Math 2270-004 Week 10 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 5.2-5.4

Mon Mar 12
- 5.2 matrix eigenspaces

Announcements:

Warm-up Exercise:

Monday Review!

We've been studying *linear transformations* $T : V \rightarrow W$ between vector spaces, which are generalizations of matrix transformations $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ given as $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$.

We've been studying how coordinates change when we change bases for finite dimensional vector spaces V.

On Friday we introduced the notion of *eigenvectors* for linear transformations $T : V \rightarrow V$, non-zero vectors $\underline{\boldsymbol{v}}$ so that T transforms $\underline{\boldsymbol{v}}$ to a multiple of itself. This multiple λ is called the *eigenvalue* of $\underline{\boldsymbol{v}}$. In other words, $T(\underline{\boldsymbol{v}}) = \lambda\,\underline{\boldsymbol{v}}$.

For our eigenvector discussion we're focusing on $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$. In this case we talk about eigenvectors of the matrix A, with eigenvalue λ, $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$. The idea is that because eigenvectors are transformed in just about the most simple way possible by the matrix, they will give us computational and conceptual insite into the matrix transformation. We'll see how this plays out.

On Friday we introduced the characteristic equation method of finding eigenvalues of a matrix first, and then the eigenvectors (eigenspace bases) second:

How to find eigenvalues and eigenvectors (eigenspace bases) systematically:

If

$$A\,\underline{\mathbf{v}} = \lambda\,\underline{\mathbf{v}}$$

$$\Leftrightarrow A\,\underline{\mathbf{v}} - \lambda\,\underline{\mathbf{v}} = \underline{\mathbf{0}}$$

$$\Leftrightarrow A\,\underline{\mathbf{v}} - \lambda\, I\,\underline{\mathbf{v}} = \underline{\mathbf{0}}$$

where I is the identity matrix.

$$\Leftrightarrow (A - \lambda\, I)\underline{\mathbf{v}} = \underline{\mathbf{0}}\,.$$

As we know, this last equation can have non-zero solutions $\underline{\mathbf{v}}$ if and only if the matrix $(A - \lambda\, I)$ is not invertible, i.e.

$$\Leftrightarrow det(A - \lambda\, I) = 0\,.$$

So, to find the eigenvalues and eigenvectors of matrix you can proceed as follows:

- Compute the polynomial in λ

$$p(\lambda) = det(A - \lambda\, I)\,.$$

If $A_{n\times n}$ then $p(\lambda)$ will be degree n. This polynomial is called the characteristic polynomial. The eigenvalues will be the roots λ_j of the characteristic equation

$$det(A - \lambda\, I) = 0.$$

- For each such root λ_j, the homogeneous solution space of vectors $\underline{\mathbf{v}}$ solving

$$\left(A - \lambda_j\, I\right)\underline{\mathbf{v}} = \underline{\mathbf{0}}$$

i.e

$$Nul\ \left(A - \lambda_j\, I\right)$$

will be the sub vector space of eigenvectors with eigenvalue λ_j. This subspace of eigenvectors will be at least one dimensional, since $\left(A - \lambda_j\, I\right)$ does not reduce to the identity and so the explicit homogeneous solutions will have free parameters. Find a basis of eigenvectors for this subspace. Follow this procedure for each eigenvalue, i.e. for each root of the characteristic polynomial.

Notation: The subspace of eigenvectors for eigenvalue λ_j is called the λ_j eigenspace, and we'll denote it by $E_{\lambda=\lambda_j}$. The basis of eigenvectors is called an eigenbasis for $E_{\lambda=\lambda_j}$.

We did part a on Friday, but didn't get to part b:

Exercise 1) a) Use the systematic algorithm to find the eigenvalues and eigenbases for the non-diagonal matrix

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

b) Use your work to describe the geometry of the linear transformation in terms of directions that get stretched:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

1a)

$$A - \lambda I = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{bmatrix}$$

$$p(\lambda) = \begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda-3)\cdot(\lambda-2) - 2 = \lambda^2 - 5\lambda + 4 = (\lambda-1)\cdot(\lambda-4).$$

So the eigenvalues of A are $\lambda = 1, 4$

$E_{\lambda=4}$: $Nul\ (A - 4\,I)$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & -2 & 0 \end{array}\right] \qquad E_{\lambda=4} = span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$$

$E_{\lambda=1}$: $Nul\ (A - I)$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 1 & 1 & 0 \end{array}\right] \qquad E_{\lambda=1} = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

Let's do part b!

b) Use the eigenspace information to describe the geometry of the linear transformation in terms of directions that get stretched.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \qquad E_{\lambda=4} = span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\} \qquad E_{\lambda=1} = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}.$$

Exercise 2) Find the eigenvalues and eigenspace bases for

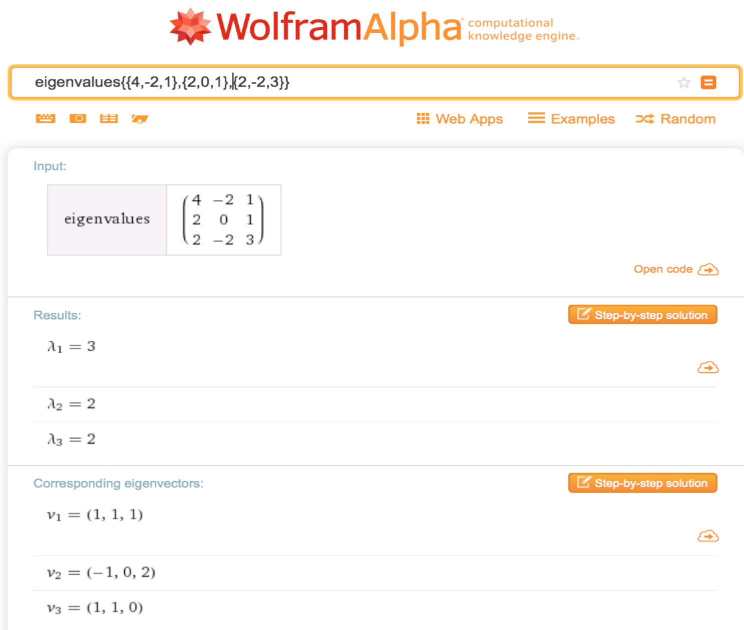
$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}.$$

(i) Find the characteristic polynomial and factor it to find the eigenvalues. ($p(\lambda) = -(\lambda - 2)^2(\lambda - 1)$)

(ii) for each eigenvalue, find bases for the corresponding eigenspaces.

(iii) Can you describe the transformation $T(\underline{\mathbf{x}}) = B\underline{\mathbf{x}}$ geometrically using the eigenbases? Does $det(B)$ have anything to do with the geometry of this transformation?

Your solution will be related to the output below:



In most of our examples so far, it turns out that by collecting bases from each eigenspace for the matrix $A_{n \times n}$, and putting them together, we get a basis for $\mathbb{R}^n$. This lets us understand the geometry of the transformation

$$T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$$

almost as well as if A is a diagonal matrix. It does not always happen that the matrix A an basis of $\mathbb{R}^n$ made consisting of eigenvectors for A. (Even when all the eigenvalues are real.) When it does happen, we say that A is diagonalizable. Here's an example of a matrix which is NOT diagonalizable:

Exercise 3: Find matrix eigenvalues and eigenspace basis for each eigenvalue, for

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}.$$

Explain why there is no basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

There are situations where we are guaranteed a basis of $\mathbb{R}^n$ made out eigenvectors of A:

<u>Theorem 1</u>: Let A be an $n \times n$ matrix with distinct real eigenvalues λ_1, λ_2 , ... λ_n. Let $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots \underline{\boldsymbol{v}}_n$ be corresponding (non-zero) eigenvectors, $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$. Then the set

$$\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \cdots \underline{\boldsymbol{v}}_n\}$$

is linearly independent, and so is a basis for $\mathbb{R}^n$.....this is one we can prove!

Theorem 2

Let $A_{n \times n}$ have factored characteristic polynomial

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \ldots (\lambda - \lambda_m)^{k_m}$$

where like terms have been collected so that each λ_j is distinct (i.e different). Notice that

$$k_1 + k_2 + \ldots + k_m = n$$

because the degree of $p(\lambda)$ is n.

- Then $1 \leq dim\left(E_{\lambda = \lambda_j}\right) \leq k_j$. If $dim\left(E_{\lambda = \lambda_j}\right) < k_j$ then the λ_j eigenspace is called defective.
- The matrix A is diagonalizable if and only if each $dim\left(E_{\lambda = \lambda_j}\right) = k_j$. In this case, one obtains an $\mathbb{R}^n$

eigenbasis simply by combining bases for each eigenspace into one collection of n vectors. (The same definitions and reasoning can apply to complex eigenvalues and eigenvectors, and a basis of $\mathbb{C}^n$.)

(The proof of this theorem is fairly involved. It was illustrated in a positive way by Exercise 2, and in a negative way by Exercise 3.)

Tues Mar 13

- 5.3 *Diagonalizable* matrices and *Similar matrices*.

Announcements:

Warm-up Exercise:

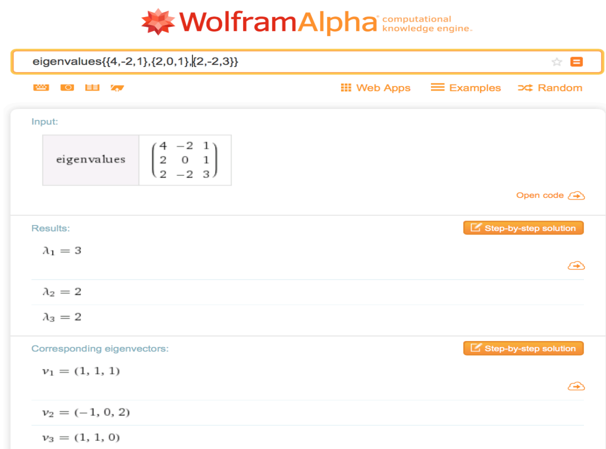
Continuing with the example from yesterday ...

If, for the matrix $A_{n \times n}$, there is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A, then   we can understand the geometry of the transformation

$$T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$$

almost as well as if A is a diagonal matrix, and so we call such matrices *diagonalizable*. Having such a basis of eigenvectors for a given matrix is also extremely useful for algebraic computations, and will give another reason for the word *diagonalizable* to describe such matrices.

Use an $\mathbb{R}^3$ basis made of out eigenvectors of the matrix B in Exercise 2, yesterday, and put them into the columns of a matrix we will call P. We could order the eigenvectors however we want, but we'll put the $E_{\lambda=2}$ basis vectors in the first two columns, and the $E_{\lambda=3}$ basis vector in the third column:



$$P := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

Now do algebra (check these steps and discuss what's going on!)

$$\begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix} \left[\begin{array}{c|c|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 4 & -4 & 3 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{array}\right] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

In other words,

$$B\,P = P\,\mathrm{D}\,,$$

where D is the diagonal matrix of eigenvalues (for the corresponding columns of eigenvectors in P).
Equivalently (multiply on the right by P^{-1} or on the left by P^{-1}):

$$B = P\,\mathrm{D}\,P^{-1} \text{ and } P^{-1}BP = \mathrm{D}.$$

<u>Exercise 1)</u> Use one of the the identities above to show how B^{100} can be computed with only two matrix multiplications!

Definition: Let $A_{n \times n}$. If there is an $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \ldots, \underline{\mathbf{v}}_n$ consisting of eigenvectors of A, then A is called diagonalizable. This is precisely why:

Write $A\,\underline{\mathbf{v}}_j = \lambda_j\,\underline{\mathbf{v}}_j$ (some of these λ_j may be the same, as in the previous example). Let P be the matrix

$$P = \left[\underline{\mathbf{v}}_1 \middle| \underline{\mathbf{v}}_2 \middle| \cdots \middle| \underline{\mathbf{v}}_n\right].$$

Then, using the various ways of understanding matrix multiplication, we see

$$A\,P = A\left[\underline{\mathbf{v}}_1 \middle| \underline{\mathbf{v}}_2 \middle| \cdots \middle| \underline{\mathbf{v}}_n\right] = \left[\lambda_1\underline{\mathbf{v}}_1 \middle| \lambda_2\underline{\mathbf{v}}_2 \middle| \cdots \middle| \lambda_n\underline{\mathbf{v}}_n\right]$$

$$= \left[\underline{\mathbf{v}}_1 \middle| \underline{\mathbf{v}}_2 \middle| \cdots \middle| \underline{\mathbf{v}}_n\right] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

$$A\,P = P\,\mathrm{D}$$

$$A = P\,\mathrm{D}\,P^{-1}$$

$$P^{-1} A\,P = \mathrm{D}\,.$$

Unfortunately, as we've already seen, not all matrices are diagonalizable:

Exercise 2) Show that

$$C := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is not diagonalizable. (Even though it has the same characteristic polynomial as B, which was diagonalizable.

Similar matrices. This generalizes the way in which diagonalizable matrices are similar to diagonal ones:

Definition The $n \times n$ matrices A, B are said to be *similar* if there is and invertible matrix P so that

$$P^{-1} A P = B.$$

Notice that *being similar* is an *equivalence relation*:

1) If A is similar to B with the matrix P, then B is similar to A, with the matrix P^{-1} :

$$P^{-1} A P = B \qquad \Rightarrow \quad A = P B P^{-1}.$$

2) A is similar to itself, with $P = I$:

$$A = I^{-1} A I$$

3) Being similar is *transitive*: if A is similar to B and B is similar to C, then A is similar to C: If we have invertible matrices P, Q so that

$$P^{-1} A P = B$$
$$Q^{-1} B Q = C$$

then

$$Q^{-1} P^{-1} A P Q = Q^{-1} B Q = C.$$

so A is similar to C via the matrix PQ.

These three "equivalence relations" mean that the space all $n \times n$ matrices can be *partitioned* into subsets of matrices which are similar to each other.

We'll see tomorrow that *similar matrices* represent the *same* linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, but with the matrices expressed with respect to different bases. For now (and for one of your homework problems tomorrow), we need to know that

Theorem Let A and B be similar matrices. Then they have the same characteristic polynomial, so the same eigenvalues. (They won't have the same eigenvectors, though.)

proof Let

$$P^{-1} A P = B.$$

Then

$$det(B - \lambda I) = det(P^{-1} A P - \lambda I)$$

$$= det \left(P^{-1} A P - \lambda P^{-1} I P\right)$$

$$= det \left(P^{-1} (A - \lambda I) P\right)$$

$$= det\left(P^{-1}\right)\, det\left(A - \lambda\, I\right)\, det\,(P)$$

$$= det\left(A - \lambda\, I\right).$$

QED

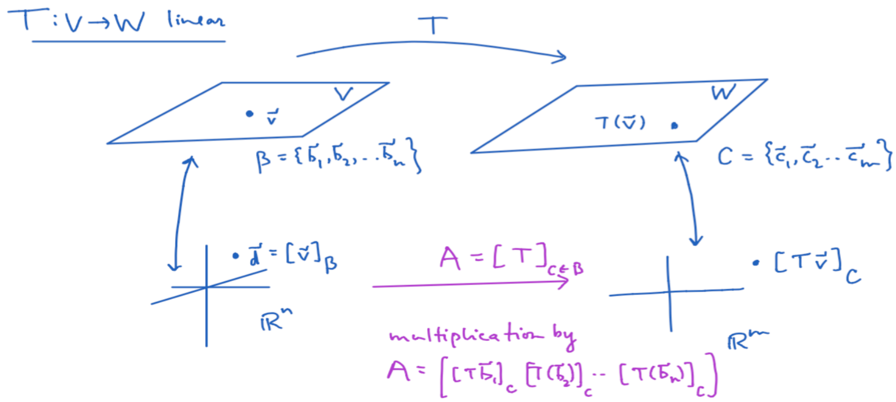
Wed Mar 14

- 5.4 Similar matrices and the matrix of a linear transformation with respect to bases

Announcements:

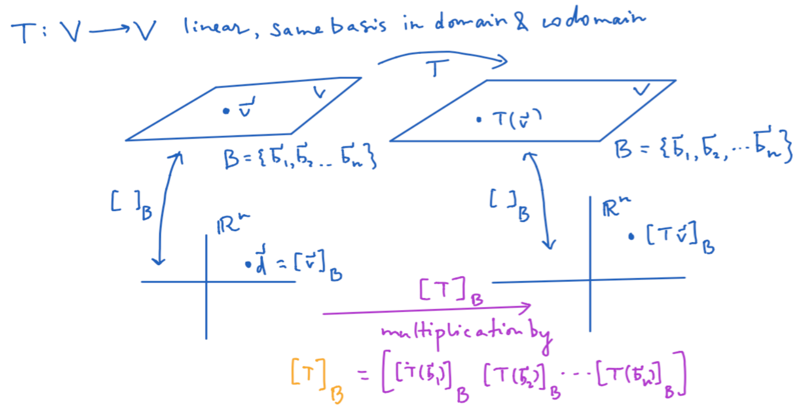
Warm-up Exercise:

If we have a linear transformation $T : V \rightarrow W$ and bases $B = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$ in V, $C = \{\underline{\boldsymbol{c}}_1, \underline{\boldsymbol{c}}_2, \ldots \underline{\boldsymbol{c}}_m\}$ in W, then the matrix of T with respect to these two bases transforms the B coordinates of vectors $\underline{\boldsymbol{v}} \in V$ to the C coordinates of $T(\underline{\boldsymbol{v}})$ in a straightforward way:

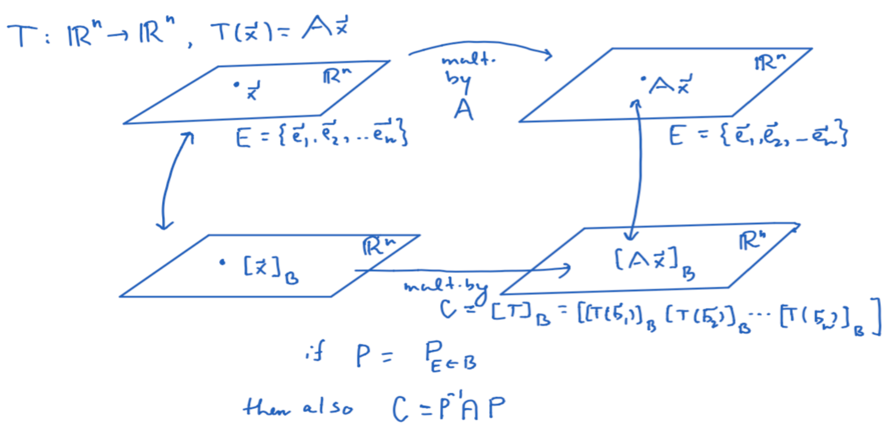


Exercise 1) Let $V = P_3 = span\{1, t, t^2, t^3\}$, $W = P_2 = span\{1, t, t^2\}$, and let $D : V \rightarrow W$ be the derivative operator. Find the matrix of D with respect to the bases $\{1, t, t^2, t^3\}$ in V and $\{1, t, t^2\}$ in W. Test your result.

A special case of the matrix for a linear transformation is when $T : V \to V$ and one uses the same basis in the domain and codomain:



And a special case of that is when $T : \mathbb{R}^n \to \mathbb{R}^n$ is a matrix transformation $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$, and we find the matrix of T with respect to a non-standard basis. This is how similar matrices arise: as descriptions of the same linear transformation, but using different bases:



A special case of similar matrices is when A is diagonalizable and P is a matrix whose columns are an eigenbasis for $\mathbb{R}^n$. Then C is a diagonal matrix with the corresponding eigenvalues in each column,