Math 2270-004 Week 1 notes

We will not necessarily finish the material from a given day's notes on that day. Or on an amazing day we may get farther than I've predicted. We may also add or subtract some material as the week progresses, but these notes represent an outline of what we will cover. These week 1 notes are for sections 1.1-1.3.

Monday January 8:

- Course Introduction
- 1.1: Systems of linear equations

• Go over course information on syllabus and course homepage:

http://www.math.utah.edu/~korevaar/2270spring18

• Note that there is a quiz this Wednesday on the material we cover today and tomorrow. Your first homework assignment will be due next Wednesday, January 17.

Then, let's begin!

• What is a linear equation? What is a system of linear equations?

any equation in some number of variables $x_1, x_2, \ldots x_n$ *which can be written in the form*

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

where b and the coefficients $a_1, a_2, \ldots a_n$ *are real or complex numbers usually known in advance.*

Question: Why do you think we call an equation like that "linear"?

Exercise 1) Which of these is a linear equation?

1a) For the variables x, y

$$3x + 4y = 6$$

1b) For the variables s, t

$$2t = 5 - \sqrt{3}\, s$$

1c) For the variables x, y

$$2x = 5 - 3\sqrt{y}\,.$$

- Then the general linear system (LS) of m equations in the n variables $x_1, x_2, \ldots x_n$ can be written as

$$\begin{aligned} a_{11}\, x_1 + a_{12}\, x_2 + \ldots + a_{1\,n}\, x_n &= b_1 \\ a_{21}\, x_1 + a_{22}\, x_2 + \ldots + a_{2\,n}\, x_n &= b_2 \\ \vdots \qquad\qquad\qquad & \vdots \quad \vdots \\ a_{m1}\, x_1 + a_{m2}\, x_2 + \ldots + a_{m\,n}\, x_n &= b_m \end{aligned}$$

In such a linear systems the coefficients $a_{i\,j}$ and the right-side number b_j are usually known. The goal is to find values for the vector $\underline{\mathbf{x}} = [x_1, x_2, \ldots x_n]$ of variables so that all equations are true. (Thus this is often called finding "simultaneous" solutions to the linear system, because all equations will be true at once.)

<u>Definition</u> *The solution set of a system of linear equations is the collection of all solution vectors to that system.*

<u>Notice</u> that we use two subscripts for the coefficients $a_{i\,j}$ and that the first one indicates which equation it appears in, and the second one indicates which variable its multiplying; in the corresponding *coefficient matrix A* , this numbering corresponds to the row and column of $a_{i\,j}$:

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1\,n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2\,n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{m\,n} \end{bmatrix}$$

Let's start small, where geometric reasoning will help us understand what's going on when we look for solutions to linear equations and to linear systems of equations.
<u>Exercise 2</u>: Describe the solution set of each single linear equation below; describe and sketch its geometric realization in the indicated Euclidean space.

<u>2a)</u> $3\,x = 5$, for $x \in \mathbb{R}$.

<u>2b)</u> $2\,x + 3\,y = 6$, for $[x, y] \in \mathbb{R}^2$.

<u>2c)</u> $2\,x + 3\,y + 4\,z = 12$, for $[x, y, z] \in \mathbb{R}^3$.

2 linear equations in 2 unknowns:

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2$$

goal: find all $[x, y]$ making both of these equations true at once. Since the solution set to each single equation is a line one geometric interpretation is that you are looking for the intersection of two lines.

Exercise 3: Consider the system of two equations E_1, E_2:

$$E_1 \qquad 5x + 3y = 1$$
$$E_2 \qquad x - 2y = 8$$

3a) Sketch the solution set in $\mathbb{R}^2$, as the point of intersection between two lines.

3b) Use the following three "elementary equation operations" to systematically reduce the system E_1, E_2 to an equivalent system (i.e. one that has the same solution set), but of the form

$$1x + 0y = c_1$$
$$0x + 1y = c_2$$

(so that the solution is $x = c_1, y = c_2$). Make sketches of the intersecting lines, at each stage.

The three types of elementary equation operation are below. Can you explain why the solution set to the modified system is the same as the solution set before you make the modification?

- interchange the order of the equations
- multiply one of the equations by a non-zero constant
- replace an equation with its sum with a multiple of a different equation.

3c) Look at your work in 3b. Notice that you could have save a lot of writing by doing this computation "synthetically", i.e. by just keeping track of the coefficients and right-side values. Using R_1, R_2 as symbols for the rows, your work might look like the computation below. Notice that when you operate synthetically the "elementary equation operations" correspond to "elementary row operations":

- interchange two rows
- multiply a row by a non-zero number
- replace a row by its sum with a multiple of another row.

$$\left[\begin{array}{cc|c} 5 & 3 & 1 \\ 1 & -2 & 8 \end{array}\right]$$

$$\begin{array}{r} R_2 \\ R_1 \end{array} \left[\begin{array}{cc|c} 1 & -2 & 8 \\ 5 & 3 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -5R_1+R_2 \end{array} \left[\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 13 & -39 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 1 & -3 \end{array}\right]$$

$$\begin{array}{r} 2R_2+R_1 \\ \\ \end{array} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right] \longrightarrow \begin{array}{l} x = 2 \\ y = -3 \end{array} \; !$$

3d) What are the possible geometric solution sets to 1, 2, 3, 4 or any number of linear equations in two unknowns?

Math 2270-004
Tues Jan 9

- 1.1 Systems of linear equations
- 1.2 Row reduction and echelon form

Announcements:

Warm-up Exercise:

Continuing our discussion of solution sets to systems of equations from yesterday:

Solutions to linear equations in 3 unknowns:

What is the geometric question we're answering in these cases?

Exercise 1) Consider the system

$$x + 2y + z = 4$$
$$3x + 8y + 7z = 20$$
$$2x + 7y + 9z = 23 .$$

Use elementary equation operations (or if you prefer, elementary row operations in the synthetic version) to find the solution set to this system. There's a systematic way to do this, which we'll talk about today. It's called Gaussian elimination. (Check Wikipedia about Gauss - he was amazing!)

Hint: The solution set is a single point, $[x, y, z] = [5, -2, 3]$.

Exercise 2) There are other possibilities. In the two systems below we kept all of the coeffients the same as in Exercise 1, except for a_{33}, and we changed the right side in the third equation, for 2a. Work out what happens in each case.

2a)

$$
\begin{aligned}
x + 2y + z &= 4 \\
3x + 8y + 7z &= 20 \\
2x + 7y + 8z &= 20\,.
\end{aligned}
$$

2b)

$$
\begin{aligned}
x + 2y + z &= 4 \\
3x + 8y + 7z &= 20 \\
2x + 7y + 8z &= 23\,.
\end{aligned}
$$

2c) What are the possible solution sets (and geometric configurations) for 1, 2, 3, 4,... equations in 3 unknowns?

Summary of the systematic method known as Gaussian elimination for solving systems of linear equations.

We write the linear system (LS) of m equations for the vector $\underline{\boldsymbol{x}} = [x_1, x_2, \ldots x_n]$ of the n unknowns as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad\quad \vdots \;\; &\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

The matrix that we get by adjoining (augmenting) the right-side $\underline{\boldsymbol{b}}$-vector to the coefficient matrix $A = [a_{ij}]$ is called the *augmented matrix* $\langle A|\underline{\boldsymbol{b}}\rangle$:

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} & b_2 \\ \vdots & & & & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} & b_m \end{array}\right]$$

Our goal is to find all the solution vectors $\underline{\boldsymbol{x}}$ to the system - i.e. the *solution set* .

There are three types of *elementary equation operations* that don't change the solution set to the linear system. They are

- interchange two of equations
- multiply one of the equations by a non-zero constant
- replace an equation with its sum with a multiple of a different equation.

And that when working with the augmented matrix $\langle A|\underline{\boldsymbol{b}}\rangle$ these correspond to the three types of *elementary row operations:*

- interchange ("swap") two rows
- multiply one of the rows by a non-zero constant
- replace a row by its sum with a multiple of a different row.

Gaussian elimination: Use elementary row operations and work column by column (from left to right) and row by row (from top to bottom) to first get the augmented matrix for an equivalent system of equations which is in

row-echelon form:

(1) All "zero" rows (having all entries $= 0$) lie beneath the non-zero rows.

(2) The leading (first) non-zero entry in each non-zero row lies strictly to the right of the one above it. These entries are called *pivots* in our text, and the corresponding columns are called *pivot columns*.

(At this stage you could "backsolve" to find all solutions.)

Next, continue but by working from bottom to top and from right to left instead, so that you end with an augmented matrix that is in

reduced row echelon form: (1),(2), together with

(3) Each leading non-zero row entry has value 1 . Such entries are called "leading 1's " and are the

"pivots" for the corresponding pivot columns.

(4) Each pivot column has $0's$ in all the entries except for the pivot entry of 1.

Finally, read off how to explicitly specify the solution set, by "backsolving" from the reduced row echelon form.

Note: There are lots of row-echelon forms for a matrix, but only one reduced row-echelon form. (We'll see why later.) All mathematical software packages have a command to find the reduced row echelon form of a matrix.

Exercise 3 Find all solutions to the system of 3 linear equations in 5 unknowns

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\ 2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 &= 7 \\ 3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 &= 27\,. \end{aligned}$$

Here's the augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array}\right]$$

Find the reduced row echelon form of this augmented matrix and then backsolve to explicitly parameterize the solution set. (Hint: it's a two-dimensional plane in $\mathbb{R}^5$, if that helps. :-))

Maple says:

```
> with(LinearAlgebra) :   # matrix and linear algebra library
> A := Matrix(3, 5, [1,-2, 3, 2, 1,
                     2,-4, 8, 3, 10,
                     3,-6, 10, 6, 5]) :
  b := Vector([10, 7, 27]) :
  <A|b>;     # the mathematical augmented matrix doesn't actually have
               # a vertical line between the end of A and the start of b
  ReducedRowEchelonForm(<A|b>);
```

$$\begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{bmatrix} \qquad (1)$$

```
> LinearSolve(A, b);
        #  this command will actually write down the general solution, using
        # Maple's way of writing free parameters, which actually makes
        # some sense.  Generally when there  are free parameters involved,
        #  there will be equivalent ways to express the solution that may
        # look different.  But usually Maple's version will look like yours,
        # because it's using the same algorithm and choosing the free parameters
        # the same way too.
```

$$\begin{bmatrix} 5 + 2\,\_t_2 - 3\,\_t_5 \\ \_t_2 \\ -3 - 2\,\_t_5 \\ 7 + 4\,\_t_5 \\ \_t_5 \end{bmatrix} \qquad (2)$$

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>
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Math 2270-004
Wed Jan 10

- 1.3 Vectors and vector equations
- Quiz today at end of class, on section 1.1-1.2 material

Announcements:

Warm-up Exercise:

1.3 Vectors and vector equations: We'll carefully define vectors, algebraic operations on vectors and geometric interpretations of these operations, in terms of displacements. These ideas will eventually give us another way to interpret systems of linear equations.

Definition: A matrix with only one column, i.e an $n \times 1$ matrix, is a *vector* in $\mathbb{R}^n$. (We also call matrices with only one row "vectors", but in this section our vectors will always be column vectors.)

Examples:

$\underline{\boldsymbol{u}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is a vector in $\mathbb{R}^2$.

$\underline{\boldsymbol{v}} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 5 \\ 0 \end{bmatrix}$ is a vector in $\mathbb{R}^5$.

- We can multiply vectors by *scalars* (real numbers) and add together vectors that are the same size. We can combine these operations.

Example: For $\underline{\boldsymbol{u}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{w}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ compute

$$\underline{\boldsymbol{u}} + 4\underline{\boldsymbol{w}} =$$

Definition: For $\underline{\boldsymbol{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \underline{\boldsymbol{v}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n; c \in \mathbb{R},$

$$\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} := \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ : \\ u_n + v_n \end{bmatrix} \quad ; \quad c\,\underline{u} := \begin{bmatrix} cu_1 \\ cu_2 \\ : \\ cu_n \end{bmatrix}.$$

<u>Exercise 1</u> Our vector notation may not be the same as what you used in math 2210 or Math 1320 or 1321. Let's discuss the notation you used, and how it corresponds to what we're doing here.

Vector addition and scalar multiplication have nice algebraic properties:

Let $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}} \in \mathbb{R}^n$, $c, d \in \mathbb{R}$. Then

(i) $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} = \underline{\boldsymbol{v}} + \underline{\boldsymbol{u}}$

(ii) $(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) + \underline{\boldsymbol{w}} = \underline{\boldsymbol{u}} + (\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}})$

(iii) $\underline{\boldsymbol{u}} + \underline{\boldsymbol{0}} = \underline{\boldsymbol{0}} + \underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$

(iv) $\underline{\boldsymbol{u}} + (-\underline{\boldsymbol{u}}) = (-\underline{\boldsymbol{u}}) + \underline{\boldsymbol{u}} = \underline{\boldsymbol{0}}$

(v) $c(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = c\,\underline{\boldsymbol{u}} + c\,\underline{\boldsymbol{v}}$

(vi) $(c + d)\underline{\boldsymbol{u}} = c\,\underline{\boldsymbol{u}} + d\,\underline{\boldsymbol{u}}$

(vii) $c(d\,\underline{\boldsymbol{u}}) = (c\,d)\,\underline{\boldsymbol{u}}$

(viii) $1\,\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$.

Exercise 2. Verify why these properties hold!

Geometric interpretation of vectors

The space $\mathbb{R}^n$ may be thought of in two equivalent ways. In both cases, $\mathbb{R}^n$ consists of all possible $n-tuples$ of numbers:

(i) We can think of those $n-tuples$ as representing points, as we're used to doing for $n = 1, 2, 3$. In this case we can write

$$\mathbb{R}^n = \{(x_1, x_2, ..., x_n), s.t.\ x_1, x_2, ..., x_n \in \mathbb{R}\}.$$

(ii) We can think of those $n-tuples$ as representing vectors that we can add and scalar multiply. In this case we can write

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_n \end{bmatrix}, s.t.\ x_1, x_2, ..., x_n \in \mathbb{R} \right\}.$$

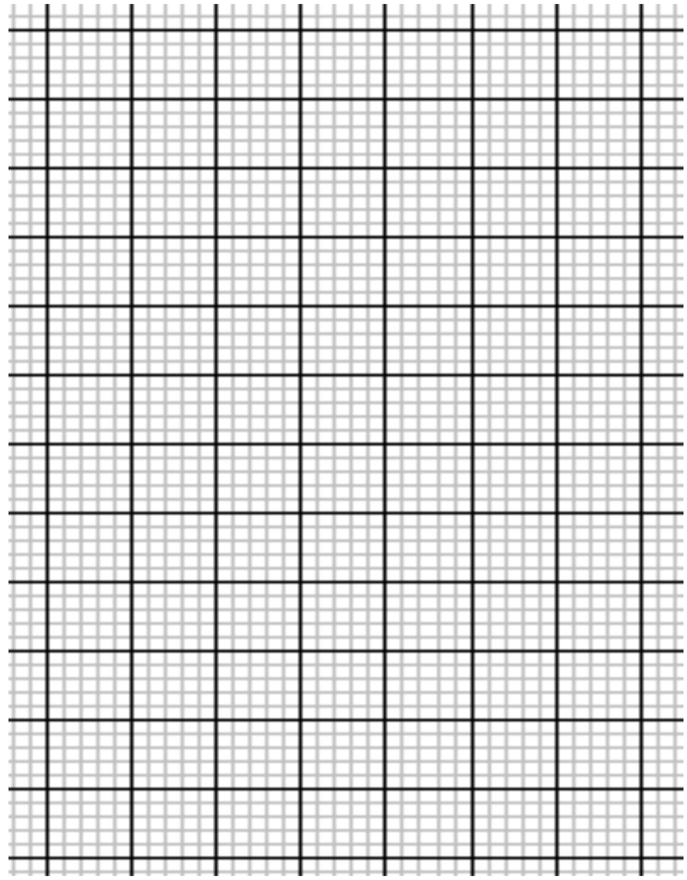
Since algebraic vectors (as above) can be used to measure geometric displacement, one can identify the two models of $\mathbb{R}^n$ as sets by identifying each point $(x_1, x_2, ... x_n)$ in the first model with the displacement vector $\underline{\boldsymbol{x}} = [x_1, x_2, ... x_m]^T$ from the origin to that point, in the second model, i.e. the position vector.

Exercise 3) Let $\underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

1a) Plot the points $(1, -1)$ and $(1, 3)$, which have position vectors $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}$. Draw these position vectors as arrows beginning at the origin and ending at the corresponding points.

1b) Compute $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}$ and then plot the point for which this is the position vector. Note that the algebraic operation of vector addition corresponds to the geometric process of composing horizontal and vertical displacements.

1c) Compute $3\ \underline{\boldsymbol{u}}$ and $-2\ \underline{\boldsymbol{v}}$ and plot the corresponding points for which these are the position vectors.



One of the key themes of this course is the idea of "linear combinations". These have an algebraic definition, as well as a geometric interpretation as combinations of displacements.

Definition: If we have a collection of n vectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_p\}$ in $\mathbb{R}^n$, then any vector $\underline{v} \in \mathbb{R}^n$ that can be expressed as a sum of scalar multiples of these vectors is called a *linear combination* of them. In other words, if we can write

$$\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_p\underline{v}_p ,$$

then $\underline{v}$ is a *linear combination* of $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the *linear combination coefficients* or *weights*.

Example You've probably seen linear combinations in previous math/physics classes, even if you didn't realize it. For example you might have expressed the position vector $\underline{\mathbf{r}}$ as a linear combination

$$\underline{\mathbf{r}} = x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\mathbf{k}}$$

where $\underline{\mathbf{i}}, \underline{\mathbf{j}}, \underline{\mathbf{k}}$ represent the unit displacements in the x, y, z directions. Since we can express these displacements using Math 2270 notation as

$$\underline{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \underline{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we have

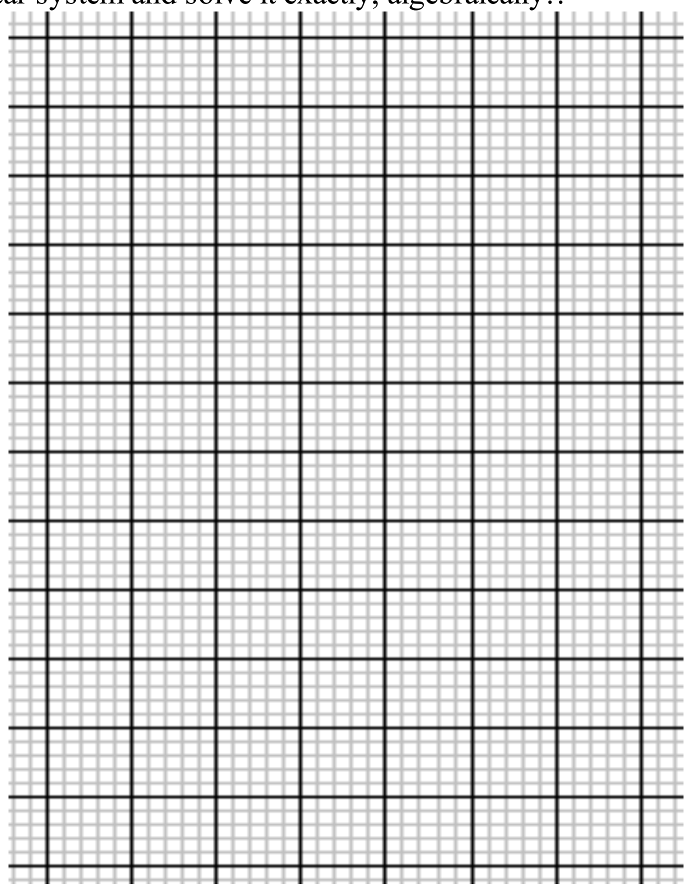
$$x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\mathbf{k}} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Exercise 4) Can you get to the point $(-2, 8) \in \mathbb{R}^2$, from the origin $(0, 0)$, by moving only in the ($\pm$) directions of $\underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$? Algebraically, this means we want to solve the linear combination problem

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

4a) Superimpose a grid related to the displacement vectors $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}$ onto the graph paper below, and, recalling that vector addition yields net displacement, and scalar multiplication yields scaled displacement, try to approximately solve the linear combination problem above, geometrically.

4b) Rewrite the linear combination problem as a linear system and solve it exactly, algebraically!!



1c) Can you get to any point (x, y) in $\mathbb{R}^2$, starting at $(0, 0)$ and moving only in directions parallel to $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}$? Argue geometrically and algebraically. How many ways are there to express $\begin{bmatrix} x \\ y \end{bmatrix}$ as a linear combination of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$?

Math 2250-004
Fri Jan 12

- 1.3 linear combinations and vector equations, continued.
- food for thought in second half of class

Announcements:

Warm-up Exercise:

1.3 Linear combinations and linear systems of equations, continued.

<u>Fundamental Fact</u> A vector equation (linear combination problem)

$$x_1\underline{\boldsymbol{v}}_1 + x_2\underline{\boldsymbol{v}}_2 + \ldots + x_p\underline{\boldsymbol{v}}_p = \underline{\boldsymbol{b}}$$

has the same solution set as the linear system whose augmented matrix is given by

$$\left[\underline{\boldsymbol{v}}_1 \;\; \underline{\boldsymbol{v}}_2 \;\; \cdots \;\; \underline{\boldsymbol{v}}_p \;\; \underline{\boldsymbol{b}}\right]$$

In particular, $\underline{\boldsymbol{b}}$ can be generated by a linear combination of $\underline{\boldsymbol{v}}_1 \;\; \underline{\boldsymbol{v}}_2 \;\; \ldots \;\; \underline{\boldsymbol{v}}_p$ if and only if there exists a solution to the linear system corresponding to the augmented matrix above.

<u>Definition:</u> The <u>span</u> of a collection of vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \;\ldots\; \underline{\boldsymbol{v}}_p$ in $\mathbb{R}^n$ is the collection of all vectors $\underline{\boldsymbol{w}}$ which can be expressed as linear combinations of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \;\ldots\; \underline{\boldsymbol{v}}_p$. We denote this collection as

$$span\{\, \underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \;\ldots\; \underline{\boldsymbol{v}}_p \} .$$

<u>Remark:</u> The mathematical meaning of the word span is related to the English meaning - as in "wing span" or "span of a bridge", but it's also different. The span of a collection of vectors goes on and on and does not "stop" at the vector or associated endpoint:

<u>Example 1)</u>

- In Exercise 4 yesterday, consider the $span\{\underline{\boldsymbol{u}}\} = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$. This is the set of all vectors of the

form $\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$ with free parameter $x_1 \in \mathbb{R}$. This is a line through the origin of $\mathbb{R}^2$ described parametrically, that we're more used to describing with implicit equation $y = -x$ (which is short for $\{\,(x, y) \in \mathbb{R}^2 \; s.t. \; y = -x\}$). (More precisely, $span\{\underline{\boldsymbol{u}}\}$ is the collection of all position vectors for that line.)

<u>Example 2</u>:

- In Exercise 1 we showed that the span of $\underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is all of $\mathbb{R}^2$.

Exercise 1) Consider the two vectors $\underline{v}_1 = [1, 0, 2]^T, \underline{v}_2 = [-1, 2, 0]^T \in \mathbb{R}^3$.

1a) Sketch these two vectors as position vectors in $\mathbb{R}^3$, using the axes below.

1b) What geometric object is $span\{\underline{v}_1\}$? Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.

1c) What geometric object is $span\{\underline{v}_1, \underline{v}_2\}$? Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.