

Math 2270-004

April 11, 2018

Applying least squares linear regression to obtain power law fits
With Maple Commands inserted at the end

A Maple "document" has text areas (like this one), and then I insert "command fields" and only do math computations in those locations so as to keep the document well organized. To insert a command field you hit the symbol that looks like a ">" sign on the left part of the upper menu, to the right of the "T". The result is...

> 3 + 4;
#do math in a region like this.... the number sign means that everything after it is just a comment and should be ignored as far as doing computations

(1)

How do you test for power laws?

Suppose you have a collection of n data points

$$[[x_1, y_1], [x_2, y_2], [x_3, y_3], \dots, [x_n, y_n]]$$

and you expect there may be a good power-law fit

$$y = b x^m$$

which approximately explains how the y_i 's are related to the x_i 's. You would like to find the "best possible" values for b and m to make this fit. It turns out, if you take the ln-ln data, your power law question is actually just a best-line fit question:

Taking (natural) logarithms of the proposed power law yields

$$\ln(y) = \ln(b) + m \ln(x).$$

So, if we write $Y = \ln(y)$ and $X = \ln(x)$, $B = \ln(b)$, this becomes the equation of a line in the new variables X and Y :

$$Y = mX + B$$

Thus, in order for there to be a power law for the original data, the ln-ln data should (approximately) satisfy the equation of a line, and vice versa. If we get a good line fit to the ln-ln data, then the slope m of this line is the power relating the original data, and the exponential e^B of the Y -intercept is the proportionality constant b in the original relation $y = b x^m$. With real data it is not too hard to see if the ln-ln data is well approximated by a line, in which case the original data is well-approximated by a power law.

Astronomical example As you may know, Isaac Newton was motivated by Kepler's (observed) Laws of planetary motion to discover the notions of velocity and acceleration, i.e. differential calculus and then integral calculus, along with the inverse square law of planetary acceleration around the sun.....from which he deduced the concepts of mass and force, and that the universal inverse square law for gravitational attraction was the ONLY force law depending only on distance between objects, which was consistent with Kepler's observations! Kepler's three observations were that

- (1) Planets orbit the sun in ellipses, with the sun at one of the ellipse foci.
- (2) A planet sweeps out equal areas from the sun, in equal time intervals, independently of where it is in its orbit.
- (3) The square of the period of a planetary orbit is directly proportional to the cube of the orbit's semi-major axis.

So, for roughly circular orbits, Keplers third law translates to the statement that the period t is related to the radius r , by the equation $t = b r^{1.5}$, for some proportionality constant b . Let's see if that's consistent with the following data:

Planet	mean distance r from sun (in astronomical units where 1=dist to earth)	Orbital period t (in earth years)
Mercury	0.387	0.241
Earth	1.	1.
Jupiter	5.20	11.86
Uranus	19.18	84.0
Pluto	39.53	248.5

Taking the (natural) logarithm of the data points, as put into a matrix, using Wolfram alpha. I entered

$Map(\ln, \{\{.387, 1, 5.20, 19.18, 39.53\}, \{.241, 1., 11.86, 84.0, 248.5\}\})$
into the command window at Wolfram alpha. (If you want to know how to something in Wolfram alpha use google, or search for a topic from the wolfram page itself).

`Map(ln,{{.387,1.,5.20,19.18,39.53},{.241,1.0,11.86,84.0,248.5}})`

Input:

$$\text{Log} / \text{e} \begin{pmatrix} 0.387 & 1 & 5.2 & 19.18 & 39.53 \\ 0.241 & 1 & 11.86 & 84 & 248.5 \end{pmatrix}$$

[Open code](#)

Result:

$$\begin{pmatrix} -0.949331 & 0 & 1.64866 & 2.95387 & 3.67706 \\ -1.42296 & 0 & 2.47317 & 4.43082 & 5.51544 \end{pmatrix}$$

Alternate form:

$$\begin{pmatrix} -0.949331 & 0 & 1.64866 & 2.95387 & 3.67706 \\ -1.42296 & 0 & 2.47317 & 4.43082 & 5.51544 \end{pmatrix}$$

We want the least squares solution to the ln-ln data, $Y = m X + B$

$$\begin{bmatrix} -.9493 & 1 \\ 0 & 1 \\ 1.64866 & 1 \\ 2.95387 & 1 \\ 3.67706 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} -1.42296 \\ 0 \\ 2.47317 \\ 4.43082 \\ 5.51544 \end{bmatrix}.$$

$$A \hat{\mathbf{x}} = \hat{\mathbf{b}}$$

$$A^T A \mathbf{x} = A^T \mathbf{b}$$

$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$$

In Maple:

```
> with(LinearAlgebra) :  
    #to load a library of Maple commands that do linear algebra computations
```

```
> A := Matrix(5, 2, [-.9493, 1, 0, 1, 1.64866, 1, 2.95387, 1, 3.67706, 1]);  
    #a 5 by 2 matrix ... then entries are given across each row and down each column
```

$$A := \begin{bmatrix} -0.9493 & 1 \\ 0 & 1 \\ 1.64866 & 1 \\ 2.95387 & 1 \\ 3.67706 & 1 \end{bmatrix} \quad (2)$$

```
> b := Vector([-1.42296, 0, 2.47317, 4.43082, 5.51544]);  
    #could also have been entered as a five by one matrix
```

$$b := \begin{bmatrix} -1.42296 \\ 0 \\ 2.47317 \\ 4.43082 \\ 5.51544 \end{bmatrix} \quad (3)$$

```
> (Transpose(A).A)^-1.Transpose(A).b;  
    #our formula for least squares solutions.
```

$$\begin{bmatrix} 1.49982355212829 \\ 0.000465682813906573 \end{bmatrix} \quad (4)$$

In Wolfram alpha: I don't know how to name objects in Wolfram alpha, which makes it somewhat cumbersome to do computations like the one above. (If anyone does know how to do this, please let me know so I can pass along the information.) But it seems to work pretty well to first write the text of the command you want into a text file (say on your laptop or a school computer), and then copy and paste that text into the Wolfram command window. Then if you need to fix problems you can fix them back in the text file. Note that in Maple and in Wolfram alpha, matrix multiplication is indicated with the period.

$A^T A$: I pasted in

$\text{Transpose}(\{\{-.9493, 1\}, \{0, 1\}, \{1.64866, 1\}, \{2.95387, 1\}, \{3.67706, 1\}\}).$
 $\{\{-.9493, 1\}, \{0, 1\}, \{1.64866, 1\}, \{2.95387, 1\}, \{3.67706, 1\}\})$

and obtained

Input interpretation:

$$\begin{pmatrix} -0.9493 & 1 \\ 0 & 1 \\ 1.64866 & 1 \\ 2.95387 & 1 \\ 3.67706 & 1 \end{pmatrix}^T \cdot \begin{pmatrix} -0.9493 & 1 \\ 0 & 1 \\ 1.64866 & 1 \\ 2.95387 & 1 \\ 3.67706 & 1 \end{pmatrix}$$

Result:

$$\begin{pmatrix} 25.8654 & 7.33029 \\ 7.33029 & 5 \end{pmatrix}$$

$A^T \underline{b}$:

$\text{Transpose}(\{\{-.9493, 1\}, \{0, 1\}, \{1.64866, 1\}, \{2.95387, 1\}, \{3.67706, 1\}\}).$
 $\{\{-1.43396\}, \{0\}, \{2.47317\}, \{4.43082\}, \{5.51544\}\}$

Input interpretation:

$$\begin{pmatrix} -0.9493 & 1 \\ 0 & 1 \\ 1.64866 & 1 \\ 2.95387 & 1 \\ 3.67706 & 1 \end{pmatrix}^T \cdot \begin{pmatrix} -1.43396 \\ 0 \\ 2.47317 \\ 4.43082 \\ 5.51544 \end{pmatrix}$$

[Open code](#)

m^T gives the transpose of m

Result:

$$\begin{pmatrix} 38.8073 \\ 10.9855 \end{pmatrix}$$

☒ Step-by-step solution

$$(A^T A)^{-1} A^T \underline{b}:$$

$$\{\{25.8645, 7.33029\}, \{7.33029, 5.\}\}^{(-1)} \cdot \{\{38.8073\}, \{10.9855\}\}$$

Input interpretation:

$$\begin{pmatrix} 25.8645 & 7.33029 \\ 7.33029 & 5 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 38.8073 \\ 10.9855 \end{pmatrix}$$

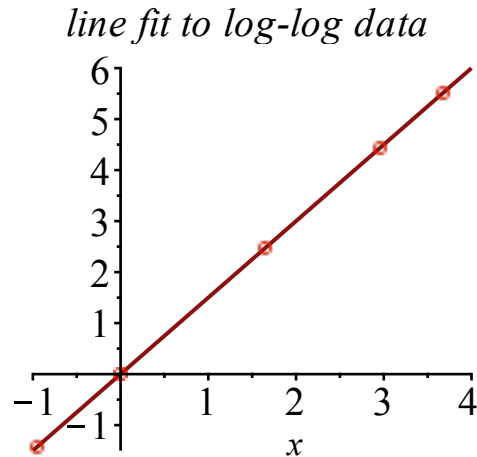
Result:

$$\begin{pmatrix} 1.50166 \\ -0.00442256 \end{pmatrix}$$

which is the same result we got from Maple, up to round-off error.

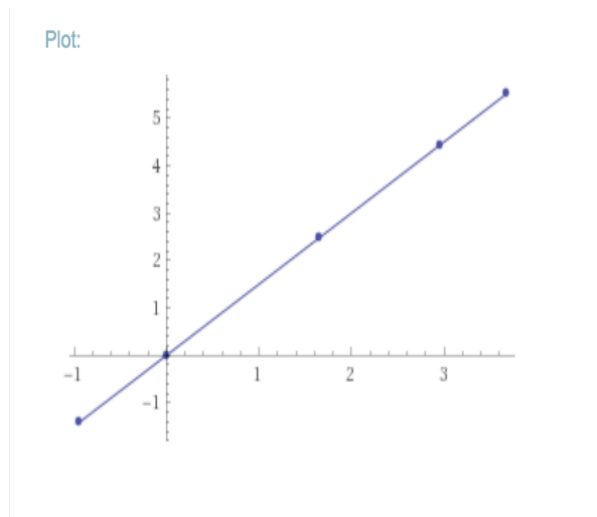
plotting the least squares best line fit, in Maple:

```
> with(plots) :  
plot1 := pointplot( { [-.9493, -1.42296], [0, 0], [1.64866, 2.47317], [2.95387, 4.43082],  
    [3.67706, 5.51544] }, color = red, symbol = circle, symbolsize = 18 ) :  
plot2 := plot(1.4998·x + .0005, x = -1 .. 4) :  
display( {plot1, plot2}, title = `line fit to log-log data`);
```



In Wolfram alpha: Wolfram alpha automatically includes the least-squares line fit to a scatter plot. Not sure whether I like this or not. :-) There are ways to get rid of the line, if you want.

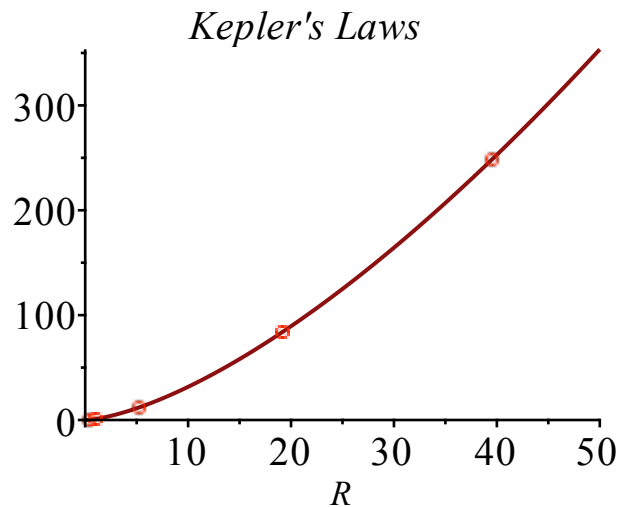
```
plot{{-.9493,-1.42296},{0,0},{1.64866,2.47317},{2.95387,4.43082},{3.67706,5.51544}}
```




```

> plot3 := pointplot( { [.387, .241], [1., 1.], [5.20, 11.86], [19.18, 84.0], [39.53, 248.5] }, color
    = red, symbol = circle, symbolsize = 18 ) :
plot4 := plot( exp(0.00046568) · R1.49982, R = 0 .. 50 ) :
display( {plot3, plot4}, title = `Kepler's Laws`);

```



In Wolfram alpha: I didn't figure out how to create the curve above at Wolfram alpha....if anyone does figure it out, let me know.

In Maple, everything at once:

```
> restart :  
> with(LinearAlgebra) :  
> Rs := Vector([.387, 1, 5.20, 19.18, 39.53]); #use shift enter to enter multiple command lines  
   Ts := Vector([.241, 1., 11.86, 84.0, 248.5]);
```

$$Rs := \begin{bmatrix} 0.387 \\ 1 \\ 5.20 \\ 19.18 \\ 39.53 \end{bmatrix}$$
$$Ts := \begin{bmatrix} 0.241 \\ 1. \\ 11.86 \\ 84.0 \\ 248.5 \end{bmatrix} \quad (5)$$

```
> lnRs := Map(ln, Rs);  
   lnTs := Map(ln, Ts);
```

$$\ln Rs := \begin{bmatrix} -0.9493305860 \\ 0 \\ 1.648658626 \\ 2.953868069 \\ 3.677059877 \end{bmatrix}$$
$$\ln Ts := \begin{bmatrix} -1.422958345 \\ 0. \\ 2.473171394 \\ 4.430816799 \\ 5.515442846 \end{bmatrix} \quad (6)$$

```
> ones := Vector([1, 1, 1, 1, 1]);  
   A := <lnRs|ones>; #make a matrix out of columns
```

$$ones := \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A := \begin{bmatrix} -0.9493305860 & 1 \\ 0 & 1 \\ 1.648658626 & 1 \\ 2.953868069 & 1 \\ 3.677059877 & 1 \end{bmatrix} \quad (7)$$

$$\begin{aligned} &> (\text{Transpose}(A).A)^{-1}.\text{Transpose}(A).\ln Ts; \text{ \#our formula for least squares solutions} \\ &\begin{bmatrix} 1.49981641316971 \\ 0.000486890692341586 \end{bmatrix} \end{aligned} \quad (8)$$