Fri Mar 29

- 5.1-5.2 Eigenvectors and eigenvalues for square matrices

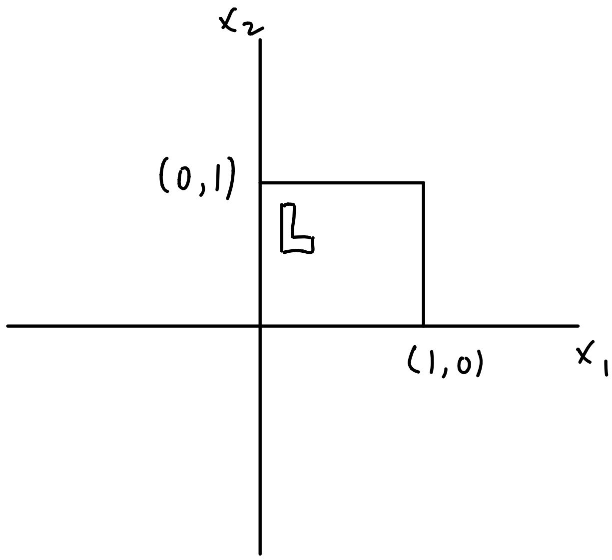
Announcements: Midterm next Friday : Thru 5.3 try for practice midterm post over weekend.

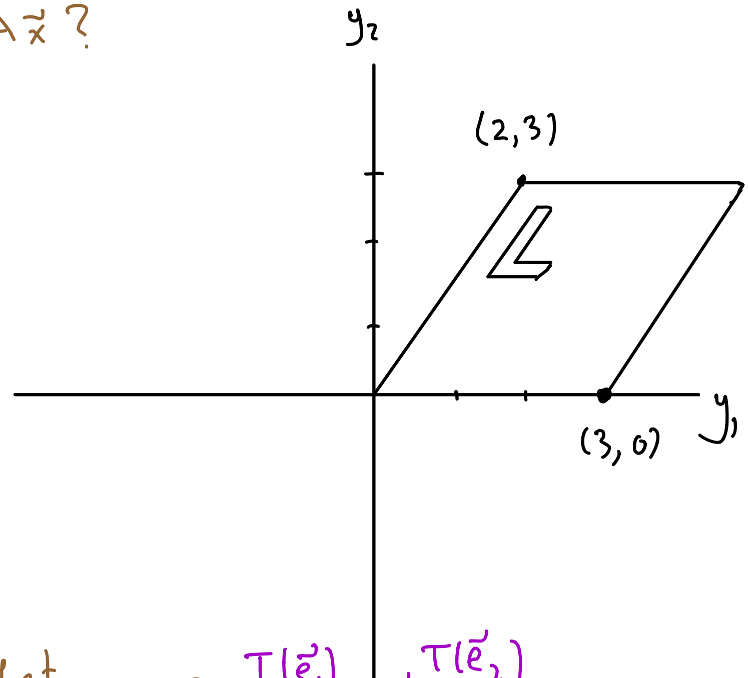
Beautiful applications after spring break

- Chptr 5: google search algorithm
- Chptr 6: orthogonality basic stats // jpeg
- Chptr 7: "Spectral theorem" stats. fft.

'til 12:57

Warm-up Exercise: a) What is the matrix for this transformation $T(\vec{x}) = A\vec{x}$?





b) How many vectors can you find that are transformed to multiples of themselves?

$T(\vec{e}_1)$ $T(\vec{e}_2)$

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$$

$T(\vec{e}_1) = 3\vec{e}_1$

$$\begin{aligned} T(x_1\vec{e}_1) &= x_1 T(\vec{e}_1) \\ &= x_1 3\vec{e}_1 \\ &= 3(x_1\vec{e}_1) \end{aligned}$$

if $T: V \to V$ is linear

and $T(\vec{v}) = \lambda\vec{v}$ $\quad \lambda \in \mathbb{R}$ $(\vec{v} \neq \vec{0})$

then $\vec{v}$ is an eigenvector

λ is called the eigenvalue

German: "eigen" means "self"

Eigenvalues and eigenvectors for square matrices.

To introduce the idea of eigenvalues and eigenvectors we'll first think geometrically.

<u>Example</u> Consider the matrix transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}. = \begin{bmatrix} 3x_1 \\ x_2 \end{bmatrix}$$

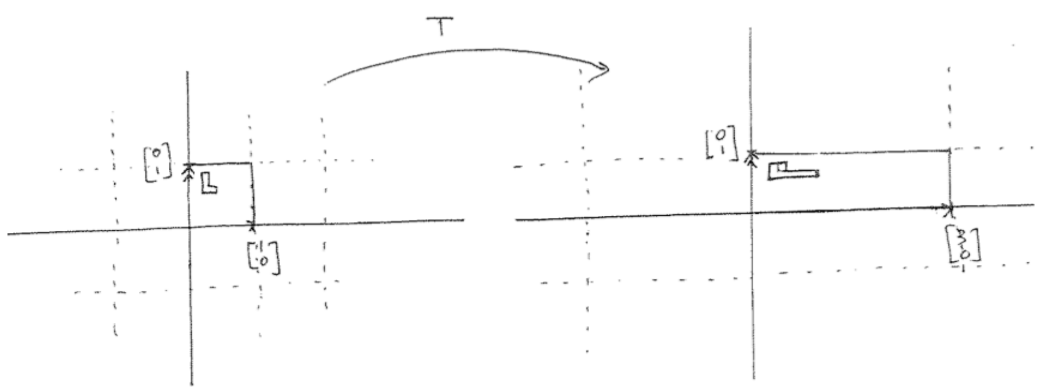
Notice that for the standard basis vectors $\underline{e}_1 = [1, 0]^T, \underline{e}_2 = [0, 1]^T$

$$T(\underline{e}_1) = 3\underline{e}_1$$
$$T(\underline{e}_2) = \underline{e}_2 \ .$$

The facts that T is linear and that it transforms $\underline{e}_1, \underline{e}_2$ by scalar multiplying them, lets us understand the geometry of this transformation completely:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T(x_1\underline{e}_1 + x_2\underline{e}_2) = x_1 T(\underline{e}_1) + x_2 T(\underline{e}_2)$$
$$= x_1(3\underline{e}_1) + x_2(1\underline{e}_2) \ .$$

In other words, T stretches by a factor of 3 in the $\underline{e}_1$ direction, and by a factor of 1 in the $\underline{e}_2$ direction, transforming a square grid in the domain into a parallel rectangular grid in the image:



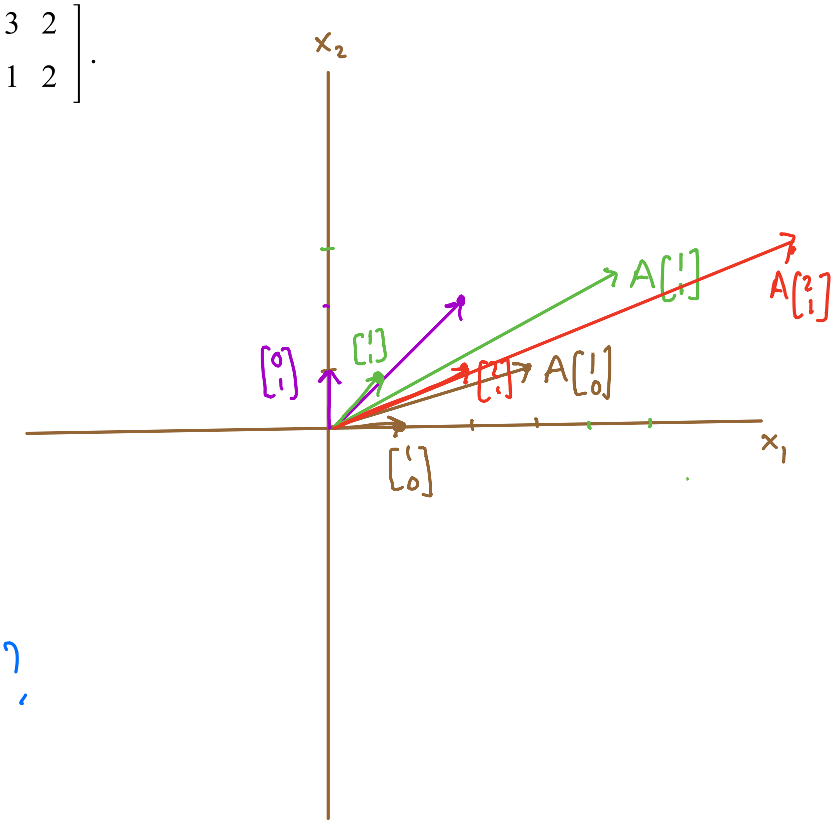
Definition: If $A_{n \times n}$ and if $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$ for a scalar λ and a vector $\underline{\boldsymbol{v}} \neq \underline{\boldsymbol{0}}$ then $\underline{\boldsymbol{v}}$ is called an eigenvector of A, and λ is called the eigenvalue of $\underline{\boldsymbol{v}}$. (In some texts the words characteristic vector and characteristic value are used as synonyms for these words.)

• In the example above, the standard basis vectors (or multiples of them) are eigenvectors, and the corresponding eigenvalues are the diagonal matrix entries. A non-diagonal matrix may still have eigenvectors and eigenvalues, and this geometric information can still be important to find. But how do you find eigenvectors and eigenvalues for non-diagonal matrices? ...

Exercise 2) Try to find eigenvectors and eigenvalues for the non-diagonal matrix, by just trying random input vectors $\underline{\boldsymbol{x}}$ and computing $A\,\underline{\boldsymbol{x}}$.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

$\vec{x}$	$A\vec{x}$	
$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}3\\1\end{bmatrix}$	
$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}2\\2\end{bmatrix}$	
$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}3&2\\1&2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$	
$\begin{bmatrix}2\\1\end{bmatrix}$	$\begin{bmatrix}8\\4\end{bmatrix}$	$\begin{bmatrix}3&2\\1&2\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}$
$\begin{bmatrix}2\\1\end{bmatrix}$	$= 4\begin{bmatrix}2\\1\end{bmatrix}$	$\lambda = 4$
$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\longleftarrow$ we want non-zero vectors
$t\begin{bmatrix}2\\1\end{bmatrix}$	$t4\begin{bmatrix}2\\1\end{bmatrix}$	are there more?



How to find eigenvalues and eigenvectors (including eigenspaces) systematically:

If

$$A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$$

$$\Leftrightarrow A\,\underline{\boldsymbol{v}} - \boldsymbol{\lambda}\,\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$$

$$\Leftrightarrow A\,\underline{\boldsymbol{v}} - \lambda\, I\,\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$$

where I is the identity matrix.

$$\Leftrightarrow (A - \lambda\, I)\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}\,.$$

I want $\vec{v} \in Nul(A-\lambda I)$
& $\vec{v} \neq \vec{0}$

As we know, this last equation can have non-zero solutions $\underline{\boldsymbol{v}}$ if and only if the matrix $(A - \lambda\, I)$ is not invertible, i.e.

need $A-\lambda I$ not reduce to I

$$\Leftrightarrow det(A - \lambda\, I) = 0\,.$$

So, to find the eigenvalues and eigenvectors of matrix you can proceed as follows:

- Compute the polynomial in λ

$$p(\lambda) = det(A - \lambda\, I)\,. = 0$$

want roots.

If $A_{n\times n}$ then $p(\lambda)$ will be degree n. This polynomial is called the characteristic polynomial of the matrix A.

- λ_j can be an eigenvalue for some non-zero eigenvector $\underline{\boldsymbol{v}}$ if and only if it's a root of the characteristic polynomial, i.e. $p(\lambda_j) = 0$. For each such root, the homogeneous solution space of vectors $\underline{\boldsymbol{v}}$ solving

$$(A - \lambda_j\, I)\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$$

will be eigenvectors with eigenvalue λ_j. This subspace of eigenvectors will be at least one dimensional, since $(A - \lambda_j\, I)$ does not reduce to the identity and so the explicit homogeneous solutions will have free parameters. Find a basis of eigenvectors for this subspace. Follow this procedure for each eigenvalue, i.e. for each root of the characteristic polynomial.

Notation: The subspace of eigenvectors for eigenvalue λ_j is called the λ_j eigenspace, and we'll denote it by $E_{\lambda=\lambda_j}$. The basis of eigenvectors is called an eigenbasis for $E_{\lambda=\lambda_j}$.

Exercise 3) a) Use the systematic algorithm to find the eigenvalues and eigenbases for the non-diagonal matrix of Exercise 2.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

b) Use your work to describe the geometry of the linear transformation in terms of directions that get stretched:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$A\vec{v} = \lambda\vec{v}$

$A\vec{v} - \lambda\vec{v} = \vec{0}$

$(A-\lambda I)\vec{v} = \vec{0}$

$\uparrow$

$|A-\lambda I| = 0$

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 2$$
$$= 6 - 5\lambda + \lambda^2 - 2$$
$$= \lambda^2 - 5\lambda + 4$$
$$= (\lambda-1)(\lambda-4) = 0$$

if $\lambda = 1$ or $\lambda = 4$

$Nul[A-\lambda I] \neq \{\vec{0}\}$.

$\lambda = 4$

$(A-4I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

basis for $E_{\lambda=4}$ is $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

long way:
$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & -2 & 0 \end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$v_1 = 2t$

$v_2 = t$

$\vec{v} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

these are the ones we found by trial and error in Exercise 2

$\lambda = 1$

$\lambda = 1$

$(A-I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$A - \lambda I = \begin{bmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{bmatrix}$

$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ✓

basis for $E_{\lambda=1} = \left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

we didn't find these, though, in Exercise 2.

Exercise 4) Find the eigenvalues and eigenspace bases for

$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}.$$

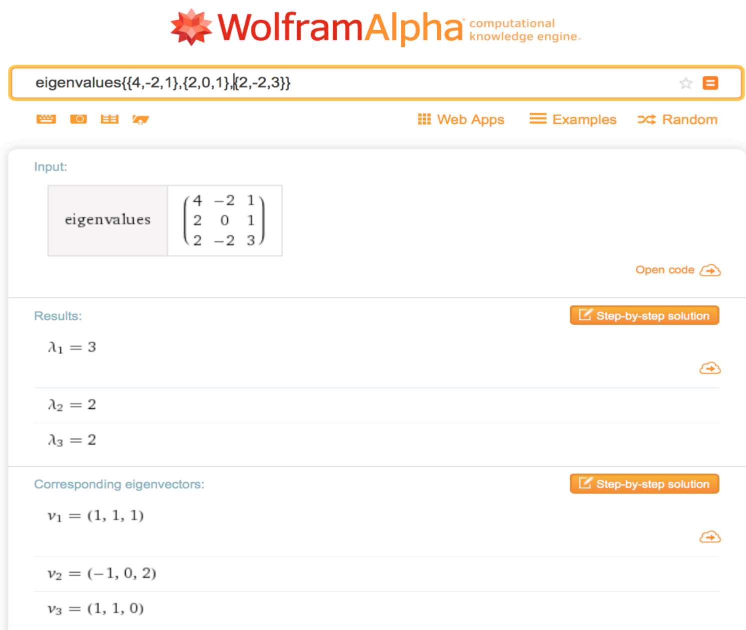
(i) Find the characteristic polynomial and factor it to find the eigenvalues.

(ii) for each eigenvalue, find bases for the corresponding eigenspaces.

(iii) Can you describe the transformation $T(\underline{\mathbf{x}}) = B\underline{\mathbf{x}}$ geometrically using the eigenbases? Does $det(B)$ have anything to do with the geometry of this transformation?

$p(\lambda) = -(\lambda-2)^2(\lambda-3)$

Your solution will be related to the output below:



In all of our examples so far, it turns out that by collecting bases from each eigenspace for the matrix $A_{n \times n}$, and putting them together, we get a basis for $\mathbb{R}^n$. This lets us understand the geometry of the transformation

$$T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$$

almost as well as if A is a diagonal matrix. This is actually something that does not always happen for a matrix A. When it does happen, we say that A is diagonalizable. Here's an example of a matrix which is NOT diagonalizable:

Exercise 5: Find matrix eigenvalues and eigenspace basis for each eigenvalue, for

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}.$$

Explain why there is no basis of $\mathbb{R}^2$ consisting of eigenvectors of A.