Wed Mar 7

- 4.7 Change of basis

Announcements: 4.7 1, 3, 7, 11, 13

- Systematize the change of coords procedure.
- quiz

'til 12:57

Warm-up Exercise: Let $V = \mathbb{P}_1 = \{ p(t) = a + bt : a, b \in \mathbb{R} \}$

Let $C = \{1, t\}$ be the "standard basis" of V

Let $B = \{1+t, 1-t\}$ another basis of V

a) If $[q]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ find $q(t)$ and $[q]_C$

b) If $[r]_C = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ find $r(t)$ and $[r]_B$

a). $[q]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \iff q(t) = 2(1+t) + 3(1-t)$

$= 5 - t$

$[5-t]_C = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

coords of an element $\vec{v}$ of a vector space with respect to a given basis $\mathcal{D} = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ are just weights when we write

$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$

$[\vec{v}]_{\mathcal{D}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

b) $[r]_C = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ $r(t) = 1 + 7t$

$[r]_C = ?$ $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$.

$1 + 7t = c_1(1+t) + c_2(1-t)$

$1 + 7t = (c_1 + c_2) + (c_1 - c_2)t$

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 7 \end{aligned} \implies \begin{aligned} c_1 &= 4 \\ c_2 &= -3 \end{aligned} \checkmark$$

The setup: Let V be a finite dimensional vector space, with two bases,

$$B = \{\underline{b}_1, \underline{b}_2, \ldots \underline{b}_n\}$$
$$C = \{\underline{c}_1, \underline{c}_2, \ldots \underline{c}_n\}$$

§4.7

How do we change from the coordinate system of the B basis to that of the C basis? If we can express the B vectors in terms of the C vectors it's straightforward:

Example Let $B = \{\underline{b}_1, \underline{b}_2\}$, $C = \{\underline{c}_1, \underline{c}_2\}$ be bases for the two-dimensional vector space V. Suppose

$$\left.\begin{aligned} \underline{b}_1 &= 4\,\underline{c}_1 + \underline{c}_2 \\ \underline{b}_2 &= -6\,\underline{c}_1 + \underline{c}_2. \end{aligned}\right\}$$

Let $[\underline{v}]_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Find $[\underline{v}]_C$.

Solution:

$$\underline{v} = x_1\,\underline{b}_1 + x_2\,\underline{b}_2$$

$$\Rightarrow [\underline{v}]_C = [x_1\,\underline{b}_1 + x_2\,\underline{b}_2]_C$$

$$= x_1\,[\underline{b}_1]_C + x_2\,[\underline{b}_2]_C$$

$$[\underline{v}]_C = \left[\,[\underline{b}_1]_C \;\; [\underline{b}_2]_C\,\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$[\underline{v}]_C = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Note that the coordinate transition matrix $\boxed{P_{C \leftarrow B}}$ would always given by

$$\left[\,[\underline{b}_1]_C \;\; [\underline{b}_2]_C\,\right].$$

no matter what the particular coordinate vectors $[\underline{b}_1]_C$, $[\underline{b}_2]_C$ are.

Exercise 1 Consider $V = \{a + b\,t\}$, the space of polynomials in t of degree ≤ 1. Let $C = \{1, t\}$ be the "standard basis". and let $B = \{\underbrace{1+t}_{\vec{b}_1},\ \underbrace{1-t}_{\vec{b}_2}\}$. Be an alternate basis.

1a) Find the transition matrix $P_{C \leftarrow B}$. $:= \begin{bmatrix} [\vec{b}_1]_C & [\vec{b}_2]_C \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$[1+t]_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$[1-t]_C = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

1b) Suppose $q(t) \in V$ with $[q]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Find $[q]_C$ by using the transition matrix $P_{C \leftarrow B}$. Compare to the direct method (which should be just as easy in this simple case).

$q(t) = 2(1+t) + 3(1-t)$
$= 5 - t$

$[q]_C = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

$[q]_C = P_{C \leftarrow B}[q]_B$
$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
$= \begin{bmatrix} 5 \\ -1 \end{bmatrix}$ ♡

1c) The transition matrix in the reverse direction must be the inverse of the original transition matrix. Find $P_{B \leftarrow C} = (P_{C \leftarrow B})^{-1}$.

$[x]_B = P_{B \leftarrow C} P_{C \leftarrow B}[\vec{x}]_B$ so $P_{B \leftarrow C} P_{C \leftarrow B} = I$

$P_{C \leftarrow B} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ so $P_{B \leftarrow C} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

1d) Suppose $r(t) = 1 + 7\,t$. Find $[r]_B$ and check your work.

$C = \{1, t\}$
$B = \{1+t, 1-t\}$

$[r]_C = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

$[r]_B = P_{B \leftarrow C}[r]_C$
$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix}$
$= \frac{1}{2}\begin{bmatrix} 8 \\ -6 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

agrees with warm-up.

(and $4(1+t) - 3(1-t) = 1 + 7t$ ✓)

Change of coordinate transition matrices work the same in every dimension.

Let V be a finite dimensional vector space, with two bases,

$$B = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ \ldots . \ \underline{\boldsymbol{b}}_n\}$$
$$C = \{\underline{\boldsymbol{c}}_1, \underline{\boldsymbol{c}}_2, \ \ldots . \ \underline{\boldsymbol{c}}_n\}.$$

Then for

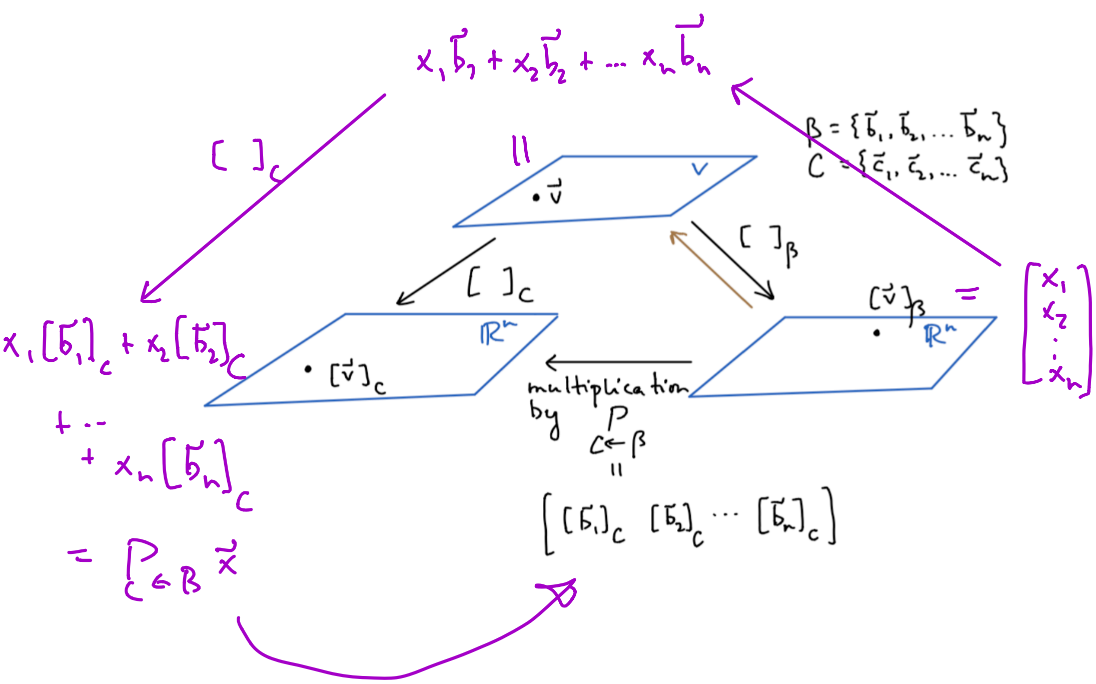
$$[\vec{v}]_B = \vec{x} \iff \underline{\boldsymbol{v}} = x_1\, \underline{\boldsymbol{b}}_1 + x_2\, \underline{\boldsymbol{b}}_2 + \ldots + x_n\, \underline{\boldsymbol{b}}_n$$

$$[\underline{\boldsymbol{v}}]_C = [\, x_1\, \underline{\boldsymbol{b}}_1 + x_2\, \underline{\boldsymbol{b}}_2 + \ldots + x_n\, \underline{\boldsymbol{b}}_n]_C$$

$$[\underline{\boldsymbol{v}}]_C = x_1\, [\underline{\boldsymbol{b}}_1]_C + x_2\, [\underline{\boldsymbol{b}}_2]_C + \ \ldots \ + x_n\, [\underline{\boldsymbol{b}}_n]_C$$

$$[\underline{\boldsymbol{v}}]_C = \big[\, [\underline{\boldsymbol{b}}_1]_C \ [\underline{\boldsymbol{b}}_2]_C \cdots [\underline{\boldsymbol{b}}_n]_C \,\big] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

$$\boxed{P_{C \leftarrow B} = \big[\, [\underline{\boldsymbol{b}}_1]_C \ [\underline{\boldsymbol{b}}_2]_C \cdots [\underline{\boldsymbol{b}}_n]_C \,\big].}$$



A special case of change of coordinates is when the vector space V is $\mathbb{R}^n$ itself. In that case there are two ways to find the coordinate transition matrices. Let

$$B = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \; .... \underline{\boldsymbol{b}}_n\}$$
$$C = \{\underline{\boldsymbol{c}}_1, \underline{\boldsymbol{c}}_2, \; .... \underline{\boldsymbol{c}}_n\}.$$

be two bases for $\mathbb{R}^n$.

<u>Method 1</u>: Let

$$E = \{\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2, \; .... \underline{\boldsymbol{e}}_n\}$$

be the standard basis. As we discussed previously and as a special case of our current discussion, since for $\underline{\boldsymbol{v}} \in \mathbb{R}^n$, $[\underline{\boldsymbol{v}}]_E = \underline{\boldsymbol{v}}$,

$$P_B = P_{E \leftarrow B} = [\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \; .... \underline{\boldsymbol{b}}_n]$$

$$P_{E \leftarrow C} = [\underline{\boldsymbol{c}}_1, \underline{\boldsymbol{c}}_2, \; .... \underline{\boldsymbol{c}}_n].$$

Since composition of matrix transformations corresponds to matrix multiplication, the transition matrix from B to C coordinates can be computed via the standard coordinate transition matrices:

$$P_{C \leftarrow B} = P_{C \leftarrow E} \, P_{E \leftarrow B} = (P_{E \leftarrow C})^{-1} \, P_{E \leftarrow B}$$

<u>Method 2:</u> Direct method. We know

$$P_{C \leftarrow B} = [[\underline{\boldsymbol{b}}_1]_C \; [\underline{\boldsymbol{b}}_2]_C \cdots [\underline{\boldsymbol{b}}_n]_C].$$

Consider the columns of the transition matrix as unknowns - as when we were finding the columns of inverses matrices by a multi-augmented matrix procedure to solve $A\,X = I$. In this case, and illustrating with $n = 2$ for simplicity,

$$P_{C \leftarrow B} = [[\underline{\boldsymbol{b}}_1]_C \; [\underline{\boldsymbol{b}}_2]_C].$$

The first column

$$[\underline{\boldsymbol{b}}_1]_C = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

satisfies

$$y_1 \underline{\boldsymbol{c}}_1 + y_2 \underline{\boldsymbol{c}}_2 = \underline{\boldsymbol{b}}_1$$

$$\left[\underline{\boldsymbol{c}}_1\ \underline{\boldsymbol{c}}_2\right]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underline{\boldsymbol{b}}_1$$

and the second column

$$\left[\underline{\boldsymbol{b}}_2\right]_C = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

satisfies

$$\left[\underline{\boldsymbol{c}}_1\ \underline{\boldsymbol{c}}_2\right]\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \underline{\boldsymbol{b}}_2$$

We solve for the two columns with a double augmented matrix reduction:

$$\left[\begin{array}{cc|cc} \underline{\boldsymbol{c}}_1 & \underline{\boldsymbol{c}}_2 & \underline{\boldsymbol{b}}_1 & \underline{\boldsymbol{b}}_2 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & y_1 & z_1 \\ 0 & 1 & y_2 & z_2 \end{array}\right]$$

$$\uparrow \qquad \uparrow$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

(And this generalizes to $\mathbb{R}^n$.)

Exercise 2 Test the two methods for finding

$$P_{C \leftarrow B}$$

where

$$B = \left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\} \qquad C = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

method 1 :

$$P_{C \leftarrow B} = P_{C \leftarrow E} \, P_{E \leftarrow B}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1.5 \\ -1 & .5 \end{bmatrix}$$

method 2 .

$$\left[\begin{array}{cc|cc} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & -1 \end{array}\right]$$

$$\begin{array}{l} \\ R_1 + R_2 \to R_2 \end{array} \left[\begin{array}{cc|cc} 1 & 1 & -3 & 2 \\ 0 & 2 & -2 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ R_2/2 \to R_2 \end{array} \left[\begin{array}{cc|cc} 1 & 1 & -3 & 2 \\ 0 & 1 & -1 & .5 \end{array}\right]$$

$$\begin{array}{l} -R_2 + R_1 \to R_1 \\ \\ \end{array} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1.5 \\ 0 & 1 & -1 & .5 \end{array}\right]$$