There is a circle of ideas related to linear independence, span, and basis for vector spaces, which it is good to try and understand carefully. That's what we'll do today. These ideas generalize (and use) ideas we've already explored more concretely, and facts we already know to be true for the vector spaces \( \mathbb{R}^n \). (A vector space that does not have a basis with a finite number of elements is said to be infinite dimensional. For example the space of all polynomials of arbitrarily high degree is an infinite dimensional vector space. We often study finite dimensional subspaces of infinite dimensional vector spaces.)

**Theorem 1** (constructing a basis from a spanning set): Let \( V \) be a vector space of dimension at least one, and let \( \text{span} \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \} = V \).

Then a subset of the spanning set is a basis for \( V \). (We followed a procedure like this to extract bases for \( \text{Col} \ A \).)

**Proof:** If \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) is not already independent,

reorder the set so that

\[
\mathbf{v}_p = d_1 \mathbf{v}_1 + d_2 \mathbf{v}_2 + \ldots + d_{p-1} \mathbf{v}_{p-1}
\]

(we know one of the vectors is a combo of the others)

so general lin. comp

\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \ldots + c_{p-1} \mathbf{v}_{p-1} + c_p \mathbf{v}_p
\]

\[
= c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \ldots + c_{p-1} \mathbf{v}_{p-1} + c_p \left( d_1 \mathbf{v}_1 + d_2 \mathbf{v}_2 + \ldots + d_{p-1} \mathbf{v}_{p-1} \right)
\]

\[
\in \text{span} \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p-1} \}
\]

continue deleting dependent vectors until remaining set is independent, with the same span you began with.

* basis *

**Theorem 2** Let \( V \) be a vector space, with basis \( \beta = \{ \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \} \). Then any set in \( V \) containing more than \( n \) elements must be linearly dependent. (We used reduced row echelon form to understand this in \( \mathbb{R}^n \).)

Let \( \alpha = \{ \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N \} \) \( N > n \)

Show \( \alpha \) is dependent:

\[
c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 + \ldots + c_N \mathbf{a}_N = \mathbf{0}
\]

Solutions?

\[
\iff \begin{bmatrix} c_1 \mathbf{a}_1 & c_2 \mathbf{a}_2 & \ldots & c_N \mathbf{a}_N \end{bmatrix} \beta = \begin{bmatrix} 0 \end{bmatrix} = \mathbf{0} \in \mathbb{R}^n
\]

\[
c_1 \mathbf{a}_1 \beta + c_2 \mathbf{a}_2 \beta + \ldots + c_N \mathbf{a}_N \beta = \mathbf{0}
\]

at least \( \frac{\text{N-\# free parameters}}{\text{N}} \) dependencies

\[
\Rightarrow \text{lots of dependencies}
\]
Theorem 3. Let \( V \) be a vector space, with basis \( \beta = \{b_1, b_2, \ldots, b_n\} \). Then no set \( \alpha = \{a_1, a_2, \ldots, a_p\} \) with \( p < n \) vectors can span \( V \). (We know this for \( \mathbb{R}^n \).)

**Proof:** Assume \( \alpha \) does span. This will lead to a contradiction. If necessary, shrink \( \alpha \) by throwing away dependent vectors, to get a basis with \( q \leq p \) elements. Then we have a basis with fewer than \( n \) vectors. 

Theorem 2 \( \implies \{b_1, b_2, \ldots, b_n\} \) is dependent.

Theorem 4. Let \( V \) be a vector space, with basis \( \beta = \{b_1, b_2, \ldots, b_n\} \). Let \( \alpha = \{a_1, a_2, \ldots, a_p\} \) be a set of independent vectors that don't span \( V \). Then \( p < n \), and additional vectors can be added to the set \( \alpha \) to create a basis \( \{a_1, a_2, \ldots, a_p, \ldots, a_n\} \) (We followed a procedure like this when we figured out all the subspaces of \( \mathbb{R}^3 \).)

**Proof:** \( \{\tilde{a}_1, \ldots, \tilde{a}_p\} \) doesn't span, but is independent. Pick \( \tilde{a}_{p+1} \in V, \tilde{a}_{p+1} \notin \text{span}\{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_p\} \).

Claim \( \{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_p, \tilde{a}_{p+1}\} \) is still independent:

**Proof:** \( c_1\tilde{a}_1 + c_2\tilde{a}_2 + \ldots + c_p\tilde{a}_p + c_{p+1}\tilde{a}_{p+1} = 0 \).

**Case 1 \( c_{p+1} = 0 \)**

If \( c_{p+1} = 0 \), then \( c_1\tilde{a}_1 + c_2\tilde{a}_2 + \ldots + c_p\tilde{a}_p = 0 \), which leads to \( c_1 = c_2 = \ldots = c_p = 0 \) because \( \{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_p\} \) is independent.

**Case 2 \( c_{p+1} \neq 0 \)**

Solve for \( \tilde{a}_{p+1} \): \( c_1\tilde{a}_1 + c_2\tilde{a}_2 + \ldots + c_p\tilde{a}_p = c_{p+1}\tilde{a}_{p+1} \). This can't happen:

- If \( c_{p+1} \neq 0 \), then \( \tilde{a}_{p+1} \notin \text{span}\{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_p\} \) is picked.
- If \( c_{p+1} = 0 \), then \( \tilde{a}_{p+1} \in \text{span}\{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_p\} \).

Both cases must be continued...
this shows $p+1 \leq n$ because I'd have $>n$ vectors, which are dependent.

$p < n$

Continue.

This ends when $p+1 = n$, because then adding another vector yields $n+1$ vectors, which must be dependent.

That means \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\} must span.

so a basis!
Theorem 5  Let $V$ be a vector space, with basis $\beta = \{b_1, b_2, \ldots b_n\}$. Then every basis for $V$ has exactly $n$ vectors. (We know this for $\mathbb{R}^n$.)

less than $n$ vectors can’t span $V$
more than $n$ vectors must be dependent.

So, dimension is well-defined
every basis has same $\# \beta$ vectors.

Theorem 6  Let $V$ be a vector space, with basis $\beta = \{b_1, b_2, \ldots b_n\}$. If $\alpha = \{a_1, a_2, \ldots a_n\}$ is another collection of exactly $n$ vectors in $V$, and if $\text{span}\{a_1, a_2, \ldots a_n\} = V$, then the set $\alpha$ is automatically linearly independent and a basis. Conversely, if the set $\{a_1, a_2, \ldots a_n\}$ is linearly independent, then $\text{span}\{a_1, a_2, \ldots a_n\} = V$ is guaranteed, and $\alpha$ is a basis. (We know all these facts for $\mathbb{R}^n$ from reduced row echelon form considerations.)

only need to check half of basis conditions, if you have right $\# \beta$ vectors.

- $\text{span}\{\tilde{a}_1, \tilde{a}_2, \ldots \tilde{a}_n\} = V$ know $\dim V = n$.
  then $\{\tilde{a}_1, \tilde{a}_2, \ldots \tilde{a}_n\}$ are independent, since else, could throw away dependent vectors to get
  a basis with $\leq n$ elements.

- If $\{\tilde{a}_1, \tilde{a}_2, \ldots \tilde{a}_n\}$ is independent it spans because if not, could add vectors not already in the span, to get a basis with $> n$ elements.
**Corollary**  Let $V$ be a vector space of dimension $n$. Then the subspaces of $V$ have dimensions 0, 1, 2, ..., $n - 1$, $n$. (We know this for $\mathbb{R}^n$.)

**Remark** We used the coordinate transformation isomorphism between a vector space $V$ with basis $\beta = \{b_1, b_2, \ldots, b_n\}$ for Theorem 2, but argued more abstractly for the other theorems. An alternate (quicker) approach is to just note that because the coordinate transformation is an isomorphism it preserves sets of independent vectors, and maps spans of vectors to spans of the image vectors, so maps subspaces to subspaces. Then every one of the theorems above follows from their special cases in $\mathbb{R}^n$, which we've already proven. But this shortcut shortchanges the conceptual ideas to some extent, which is why we've discussed the proofs more abstractly.
Tues Mar 6
- 4.6 The four subspaces associated with a matrix. the rank of a matrix.

Announcements:

Warm-up Exercise:

\[
\begin{align*}
\begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} & \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} \\
\end{align*}
\]

```
10:57 "take two" Let 
\[
A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}
\]
```

Find bases for

in domain of 
\[
0 \\
T(\vec{x}) = A\vec{x}
\]

\[
\begin{align*}
\text{Null } A & = \{ \begin{bmatrix} -1 \end{bmatrix} \} \\
\text{Row } A & = \{ \begin{bmatrix} 1 \end{bmatrix} \}
\end{align*}
\]

null vectors are the weights of column dependencies
- \( \vec{a}_1 + \vec{a}_2 = \vec{0} \)
- \(-1.\vec{a}_1 + \vec{a}_2 = \vec{0} \)

\[
\begin{bmatrix} -1 \end{bmatrix} \text{ is null vectr}
\]

in domain of 
\[
\text{Col } A = \text{row } (A^T) = \{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \}
\]

\[
\begin{align*}
\text{Null } A^T & = \{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \} \\
-2 \vec{b}_1 + \vec{b}_2 & = \vec{0}
\end{align*}
\]
Let $A$ be an $m \times n$ matrix. There are four subspaces associated with $A$. To keep them straight, keep in mind the associated linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^m \text{ given by } T(x) = A x.$$ 

And, as usual, we can express $A$ in terms of its columns, $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \ldots \ \mathbf{a}_n]$. Then the two subspaces we know well are

$$\text{Col } A = \text{span} \{ \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \} \subseteq \mathbb{R}^m$$

$$\text{Nul } A = \{ \mathbf{x} \in \mathbb{R}^n : A \mathbf{x} = \mathbf{0} \} \subseteq \mathbb{R}^n.$$ 

And, in your homework you already figured out the "rank + nullity" theorem, that

$$\dim (\text{Col } A) + \dim (\text{Nul } A) = n.$$ 

The reason for this is that if $p$ is the number of pivots in the reduced row echelon form of $A$, then

$$\dim (\text{Col } A) = p$$
$$\dim (\text{Nul } A) = n - p.$$ 

The number of pivots, i.e. $\dim (\text{Col } A)$ is called the rank of the matrix $A$. What are the other two subspaces and why do we care? Well,
First, recall the geometry fact that the dot product of two vectors in \( \mathbb{R}^n \) is zero if and only if the vectors are perpendicular, i.e.

\[
\mathbf{u} \cdot \mathbf{v} = 0 \quad \text{if and only if} \quad \mathbf{u} \perp \mathbf{v}.
\]

(Well, we really only know this in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) so far, from multivariable Calculus class. But it's true for all \( \mathbb{R}^n \), as we'll see in Chapter 6.) So for a vector \( \mathbf{x} \in \text{Nul } A \) we can interpret the equation 

\[
A \mathbf{x} = \mathbf{0}
\]
as saying that \( \mathbf{x} \) is perpendicular to every row of \( A \). Because the dot product distributes over addition, we see that each \( \mathbf{x} \in \text{Nul } A \) is perpendicular to every linear combination of the rows of \( A \). This motivates the next subspace associated with \( A \), namely the rowspace. In other words, if we express \( A \) in terms of its rows,

\[
A = \begin{bmatrix}
R_1 \\
R_2 \\
\vdots \\
R_m
\end{bmatrix}
\]

then

\[
\text{Row } A := \text{span} \{ R_1, R_2, \ldots, R_m \} \subseteq \mathbb{R}^n.
\]

As we do elementary operations on the rows of \( A \) we don't change their span, so we get a great basis for \( \text{Row } A \) by using the non-zero rows of \( \text{rref } (A) \)...as in your food for thought this past Friday, and this week's homework. So, the dimension of \( \text{Row } (A) \) is \( p \), the number of pivots in the reduced matrix. So in the domain \( \mathbb{R}^n \), we have this picture:

\[
dim(\text{Nul } A) = n - p
\]

\[
dim(\text{Row } A) = p
\]

\[
\text{Nul } A \perp \text{Row } A.
\]

The final subspace lives in the codomain \( \mathbb{R}^m \), along with \( \text{Col } A \). Well, \( \text{Col } A = \text{Row } A^T \). And so \( \text{Nul } A^T \) is the final subspace. Since \( A^T \) has \( m \) columns an \( p \) pivots, there are \( m - p \) free parameters when we solve \( A^T \mathbf{x} = 0 \), so \( \dim(\text{Nul } A^T) = m - p \) and in the codomain \( \mathbb{R}^m \) we have this picture:

\[
\text{Col } A = \text{Row } (A^T)
\]

\[
dim(\text{Row } A^T) = p
\]

\[
dim(\text{Nul } A^T) = m - p
\]

\[
\text{Nul } A^T \perp \text{Row } A^T.
\]
small example.

\[ T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]

\[
T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}
\]

\[ T: \text{row } A \rightarrow \text{col } A \text{ is an isomorphism} \]

\[ \text{row } A = \text{span} \{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \} \]

\[ \text{col } A = \text{span} \{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \} \]

\[ \text{nul } A = \text{span} \{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \} \]

\[ \text{nul } A^T = \text{span} \{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \} \]

\[ \text{image of } \text{nul } A \]

\[ (2, 4) \text{ in col } A \]
Here's a schematic of what's going on, stolen from the internet. The web site I stole it from looks pretty good....

http://www.itsshared.org/2015/06/the-four-fundamental-subspaces.html
More details on the decompositions .... In the domain $\mathbb{R}^n$, the two subspaces associated to $A$ are $Row A$ and $Nul A$. Notice that the only vector in their intersection is the zero vector, since

$$ \mathbf{x} \in Row A \cap Nul A \quad \Rightarrow \quad \mathbf{x} \cdot \mathbf{x} = 0 \quad \Rightarrow \quad \mathbf{x} = \mathbf{0}. $$

So, let

$$ \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\} \quad \text{be a basis for} \quad Row A $$

$$ \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-p}\} \quad \text{be a basis for} \quad Nul A. $$

Then we can check that set of $n$ vectors obtained by taking the union of the two sets,

$$ \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-p}\} $$

is actually a basis for $\mathbb{R}^n$. This is because we can show that the $n$ vectors in the set are linearly independent, so they automatically span $\mathbb{R}^n$ and are a basis: To check independence, let

$$ c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \ldots + c_p \mathbf{u}_p + d_1 \mathbf{v}_1 + d_2 \mathbf{v}_2 + \ldots + d_{n-p} \mathbf{v}_{n-p} = \mathbf{0}. $$

then

$$ c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \ldots + c_p \mathbf{u}_p = -d_1 \mathbf{v}_1 - d_2 \mathbf{v}_2 - \ldots - d_{n-p} \mathbf{v}_{n-p}. $$

Since the vector on the left is in $Row A$ and the one that it equals on the right is in $Nul A$, this vector is the zero vector:

$$ c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \ldots + c_p \mathbf{u}_p = 0 = -d_1 \mathbf{v}_1 - d_2 \mathbf{v}_2 - \ldots - d_{n-p} \mathbf{v}_{n-p}. $$

Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-p}\}$ are linearly independent sets, we deduce from these two equations that

$$ c_1 = c_2 = \ldots = c_p = 0, \quad d_1 = d_2 = \ldots = d_{n-p} = 0. $$

Q.E.D.

So the picture on the previous page is completely general, also for the decomposition of the codomain. One can check that the transformation $T(\mathbf{x}) = A \mathbf{x}$ restricts to an isomorphism from $Row A$ to $Col A$, because it is $1-1$ on these subspaces of equal dimension, so must also be onto. So, $T$ squashes $Nul A$, and maps every translation of $Nul A$ to a point in $Col A$. More precisely, Each

$$ \mathbf{x} \in \mathbb{R}^n $$

can be written uniquely as

$$ \mathbf{x} = \mathbf{u} + \mathbf{v} \quad \text{with} \quad \mathbf{u} \in Row A, \quad \mathbf{v} \in Nul A. $$

and

$$ T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = T(\mathbf{u}) \in Col (A). $$

As sets,

$$ T(\{\mathbf{u} + Nul A\}) = T(\mathbf{u}). $$