There is a circle of ideas related to linear independence, span, and basis for vector spaces, which it is good to try and understand carefully. That's what we'll do today. These ideas generalize (and use) ideas we've already explored more concretely, and facts we already know to be true for the vector spaces $\mathbb{R}^n$. (A vector space that does not have a basis with a finite number of elements is said to be *infinite dimensional*. For example the space of all polynomials of arbitrarily high degree is an infinite dimensional vector space. We often study finite dimensional subspaces of infinite dimensional vector spaces.)

Theorem 1 (constructing a basis from a spanning set): Let V be a vector space of dimension at least one, and let $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\} = V$.
Then a subset of the spanning set is a basis for V. (We followed a procedure like this to extract bases for *Col A*.)

proof: If $\{\vec{v}_1, \ldots \vec{v}_p\}$ is not already independent,
reorder the set so that

$$\vec{v}_p = d_1\vec{v}_1 + d_2\vec{v}_2 + \ldots + d_{p-1}\vec{v}_{p-1}$$

(we know one of the vectors is a combo of the others)

so general lin. comb

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_{p-1}\vec{v}_{p-1} + c_p\vec{v}_p$$
$$= c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_{p-1}\vec{v}_{p-1} + c_p(d_1\vec{v}_1 + d_2\vec{v}_2 + \ldots + d_{p-1}\vec{v}_{p-1})$$
$$\in span\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_{p-1}\}$$

continue deleting dependent vectors until remaining set is independent, with the same span you began with

* basis *

Theorem 2 Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$. Then any set in V containing more than n elements must be linearly dependent. (We used reduced row echelon form to understand this in $\mathbb{R}^n$.)

Let $\alpha = \{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_N\}$ $\quad N > n$

Show α is dependent:

$$c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2 + \ldots + c_N\vec{\alpha}_N = \vec{0} \qquad \text{Solutions?}$$

$$\Longleftrightarrow \left[c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2 + \ldots + c_N\vec{\alpha}_N\right]_\beta = [\vec{0}]_\beta = \vec{0} \in \mathbb{R}^n$$

$$c_1[\vec{\alpha}_1]_\beta + c_2[\vec{\alpha}_2]_\beta + \ldots + c_N[\alpha_N]_\beta = \vec{0}$$

$$n\left\{ \underbrace{\left[[\alpha_1]_\beta \ [\vec{\alpha}_2]_\beta \ \cdots \ [\alpha_N]_\beta \right]}_{N} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = \vec{0} \right.$$

at least $N-n$ free parameters
$\Rightarrow$ lots of dependencies

Theorem 3 Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$. Then no set $\alpha = \{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_p\}$ with $p < n$ vectors can span V. (We know this for $\mathbb{R}^n$.)

proof: Assume α does span. This will lead to a contradiction.

$\Downarrow$

if necessary shrink α by throwing away dependent vectors, to get a basis with $q \le p$ elements

Then we have a basis with fewer than n vectors

Theorem 2 $\Rightarrow \{\vec{b}_1, \vec{b}_2 \ldots \vec{b}_n\}$ is dependent $\Rightarrow\!\Leftarrow$.

Theorem 4 Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$. Let $\alpha = \{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_p\}$ be a set of independent vectors that don't span V. Then $p < n$, and additional vectors can be added to the set α to create a basis $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_p, \ldots \underline{\boldsymbol{a}}_n\}$ (We followed a procedure like this when we figured out all the subspaces of $\mathbb{R}^3$.)

proof: $\{\vec{a}_1, \ldots \vec{a}_p\}$ doesn't span, but is independent

pick $\vec{a}_{p+1} \in V$, $\vec{a}_{p+1} \notin \text{span}\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_p\}$.

Claim $\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_p, \vec{a}_{p+1}\}$ is still independent

pf $\quad c_1\vec{a}_1 + c_2\vec{a}_2 + \ldots + c_p\vec{a}_p + c_{p+1}\vec{a}_{p+1} = \vec{0}$.

Case 1 $\quad c_{p+1} = 0$

Case 2 $\quad c_{p+1} \ne 0$

to be continued...

Case 1 if $c_{p+1} = 0$

then $c_1\vec{a}_1 + c_2\vec{a}_2 + \ldots + c_p\vec{a}_p = \vec{0}$

$\Rightarrow c_1 = c_2 = \ldots c_p = 0$

because $\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_p\}$ independent

Case 2 $\quad c_{p+1} \ne 0 \Longrightarrow$ solve for $\vec{a}_{p+1}$: $\quad c_1\vec{a}_1 + c_2\vec{a}_2 + \ldots + c_p\vec{a}_p = -c_{p+1}\vec{a}_{p+1}$

$$-\frac{c_1}{c_{p+1}}\vec{a}_1 - \frac{c_2}{c_{p+1}}\vec{a}_2 - \ldots - \frac{c_p}{c_{p+1}}\vec{a}_p = \vec{a}_{p+1}$$

can't happen: shows $\vec{a}_{p+1} \in \text{span}\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_p\}$

picked $\vec{a}_{p+1} \notin \text{span}\{\vec{a}_1, \ldots \vec{a}_p\}$

this shows $p+1 \leq n$ because I'd have $> n$ vectors,
which are dependent

$p < n$.

Continue.

This ends when $p+1 = n$, because then adding another vector yields $n+1$ vectors, which must be dependent.

That means $\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_n\}$ must span.

So a basis! $\blacksquare$

Theorem 5 Let Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$. Then every basis for V has exactly n vectors. (We know this for $\mathbb{R}^n$.)

less than n vectors can't span V

more than n vectors must be dependent.

So, dimension is well-defined

every basis has same # of vectors.

Theorem 6 Let Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$. If $\alpha = \{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n\}$ is another collection of exactly n vectors in V, and if $span\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n\} = V$, then the set α is automatically linearly independent and a basis. Conversely, if the set $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n\}$ is linearly independent, then $span\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n\} = V$ is guaranteed, and α is a basis. (We know all these facts for $\mathbb{R}^n$ from reduced row echelon form considerations.)

only need to check half of basis conditions, if you have right # of vectors.

- $span\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_n\} = V$ know $\dim V = n$.
  then $\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_n\}$ are independent, since else, could throw away dependent vectors to get a basis with $< n$ elements.
- If $\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_n\}$ is independent
  it spans because if not, could add vectors not already in the span, to get a basis with $> n$ elements.

Corollary Let Let V be a vector space of dimension n. Then the subspaces of V have dimensions $0, 1, 2, \ldots n-1, n$. (We know this for $\mathbb{R}^n$.)

$\dim\{\vec{0}\}$.

proof Build a basis for your subspace W
(if $W \neq \{\vec{0}\}$, pick $\vec{a}_1 \neq \vec{0}$.
then keep adding vectors not in the span of what you have, to get a basis for W)
# of vectors in basis is $\leq n$,
since $\geq n+1$ vectors would be dependent

Remark We used the coordinate transformation isomorphism between a vector space V with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$ for Theorem 2, but argued more abstractly for the other theorems. An alternate (quicker) approach is to just note that because the coordinate transformation is an isomorphism it preserves sets of independent vectors, and maps spans of vectors to spans of the image vectors, so maps subspaces to subspaces. Then every one of the theorems above follows from their special cases in $\mathbb{R}^n$, which we've already proven. But this shortcut shortchanges the conceptual ideas to some extent, which is why we've discussed the proofs more abstractly.

Tues Mar 6

- 4.6 The four subspaces associated with a matrix. the rank of a matrix.

Announcements:

'f'l 10:57

Warm-up Exercise: "take two" Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = [\vec{a}_1\ \vec{a}_2]$, $A^T = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = [\vec{b}_1\ \vec{b}_2]$

Find bases for

in domain of $T(\vec{x}) = A\vec{x}$:

$\text{Nul } A = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

null vectors are the weights of column dependencies

$-\vec{a}_1 + \vec{a}_2 = \vec{0}$

$-1 \cdot \vec{a}_1 + 1 \cdot \vec{a}_2 = \vec{0}$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is null vector

$\text{Row } A = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

in domain of $S(\vec{x}) = A^T\vec{x}$ (codomain of T):

$\text{Col } A = \text{row}(A^T) = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$\text{Nul } A^T = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ $-2\vec{b}_1 + 1\vec{b}_2 = \vec{0}$

Let A be an $m \times n$ matrix. There are four subspaces associated with A. To keep them straight, keep in mind the associated linear transformation

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ given by } T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}.$$

And, as usual, we can express A in terms of its columns, $A = \left[\underline{\boldsymbol{a}}_1\ \underline{\boldsymbol{a}}_2\ \ \ldots\ \underline{\boldsymbol{a}}_n\ \right]$. Then the two subspaces we know well are

$$Col\,A = span\left\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2,\ \ldots\ \underline{\boldsymbol{a}}_n\right\} \subseteq \mathbb{R}^m$$

$$Nul\,A = \{\underline{\boldsymbol{x}} \in \mathbb{R}^n : A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}\} \subseteq \mathbb{R}^n.$$

And, in your homework you already figured out the "rank + nullity" theorem, that

$$dim\,(Col\,A) + dim\,(Nul\,A) = n. \quad = \#\text{ columns} = \text{dim domain}$$

The reason for this is that if p is the number of pivots in the reduced row echelon form of A, then

$$dim\,(Col\,A) = p$$
$$dim\,(Nul\,A) = n - p.$$

The number of pivots, i.e. $dim\,(Col\,A)$ is called the *rank* of the matrix A. What are the other two subspaces and why do we care? Well,

- First, recall the geometry fact that the dot product of two vectors in $\mathbb{R}^n$ is zero if and only if the vectors are perpendicular, i.e.

$$\underline{\boldsymbol{u}} \cdot \underline{\boldsymbol{v}} = 0 \quad \text{if and only if } \underline{\boldsymbol{u}} \perp \underline{\boldsymbol{v}}.$$

(Well, we really only know this in $\mathbb{R}^2$ or $\mathbb{R}^3$ so far, from multivariable Calculus class. But it's true for all $\mathbb{R}^n$, as we'll see in Chapter 6.) So for a vector $\underline{\boldsymbol{x}} \in Nul\,A$ we can interpret the equation

$$A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$$

as saying that $\underline{\boldsymbol{x}}$ is perpendicular to every row of A. Because the dot product distributes over addition, we see that each $\underline{\boldsymbol{x}} \in Nul\,A$ is perpendicular to every linear combination of the rows of A. This motivates the next subspace associated with A, namely the rowspace. In other words, if we express A in terms of its rows,

$$A = \begin{bmatrix} \underline{\boldsymbol{R}}_1 \\ \underline{\boldsymbol{R}}_2 \\ \vdots \\ \underline{\boldsymbol{R}}_m \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} \vec{R}_1 \cdot \vec{x} \\ \vec{R}_2 \cdot \vec{x} \\ \vdots \\ \vec{R}_m \cdot \vec{x} \end{bmatrix}$$

if $A\vec{x} = \vec{0}$

$\vec{x} \perp$ each row of A

then

$$Row\,A := span\{\underline{\boldsymbol{R}}_1, \underline{\boldsymbol{R}}_2, \ldots \underline{\boldsymbol{R}}_m\} \subseteq \mathbb{R}^n.$$

And, $Row\,A \perp Nul\,A$.

As we do elementary operations on the rows of A we don't change their span, so we get a great basis for $Row\,A$ by using the non-zero rows of $rref(A)$...as in your food for thought this past Friday, and this week's homework. So, the dimension of $Row(A)$ is p, the number of pivots in the reduced matrix. So in the domain $\mathbb{R}^n$, we have this picture:

$$dim(Nul\,A) = n - p$$
$$dim(Row\,A) = p$$
$$Nul\,A \perp Row\,A.$$

The final subspace lives in the codomain $\mathbb{R}^m$, along with $Col\,A$. Well, $Col\,A = Row\,A^T$. And so $Nul\,A^T$ is the final subspace. Since A^T has m columns an p pivots, there are $m - p$ free parameters when we solve $A^T\underline{\boldsymbol{y}} = 0$, so $dim\left(Nul\,A^T\right) = m - p$ and in the codomain $\mathbb{R}^m$ we have this picture:

$$Col\,A = Row\left(A^T\right)$$
$$dim\left(Row\,A^T\right) = p$$
$$dim\left(Nul\,A^T\right) = m - p$$
$$Nul\,A^T \perp Row\,A^T.$$

small example.

$T:\mathbb{R}^2\to\mathbb{R}^2$

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}1&1\\2&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = A\vec{x}$$

$$S\left(\begin{bmatrix}y_1\\y_2\end{bmatrix}\right)=\begin{bmatrix}1&2\\1&2\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}$$

$$\begin{bmatrix}1&1\\2&2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$$

T : rows $A \to$ col A is an isomorphism

$A\vec{x}=\vec{0}$

$\begin{bmatrix}\vec{R}_1\\ \vec{R}_2\end{bmatrix}\vec{x}=\vec{0}$

$\vec{R}_1\cdot\vec{x}=0$

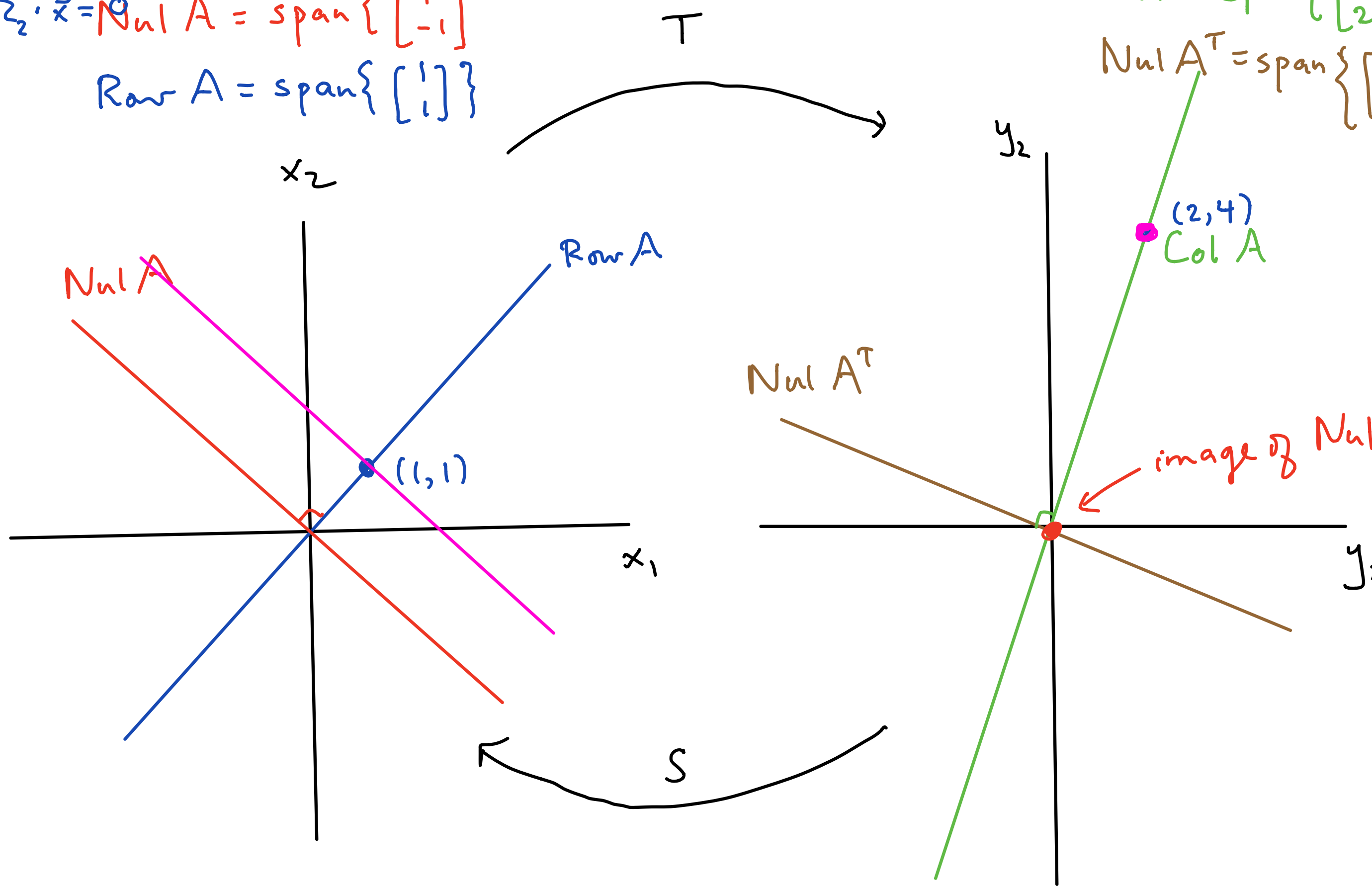
$\vec{R}_2\cdot\vec{x}=0$

Nul A = span$\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$

Row A = span$\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$

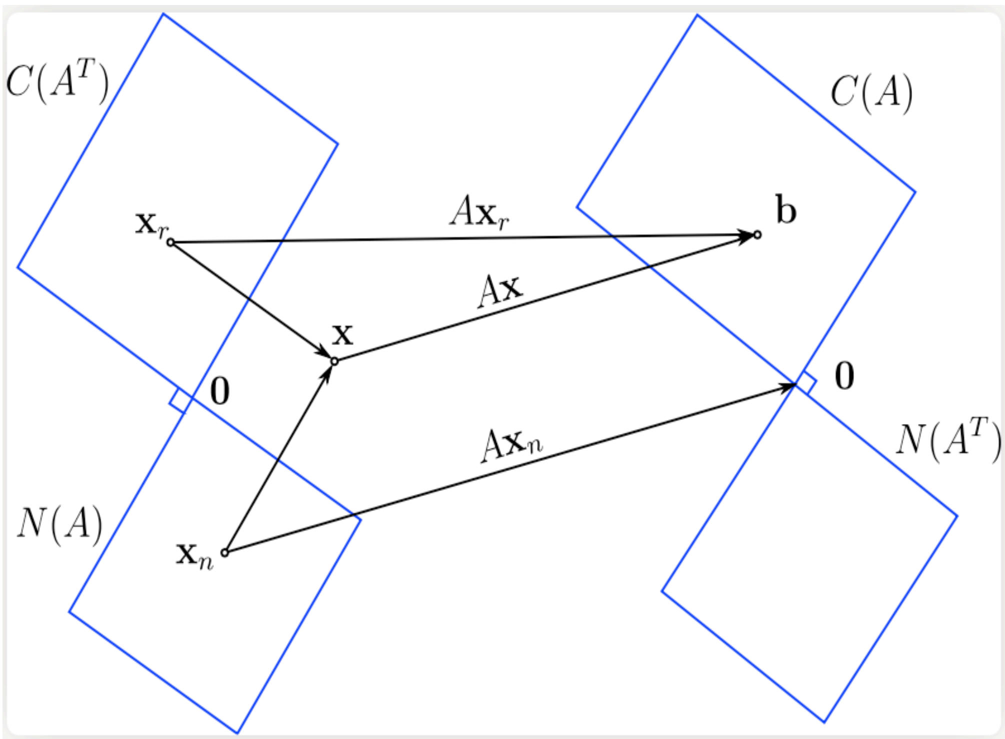
Col A = span$\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$

Nul A^T = span$\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$



Here's a schematic of what's going on, stolen from the internet. The web site I stole it from looks pretty good....

http://www.itshared.org/2015/06/the-four-fundamental-subspaces.html



More details on the decompositions .... In the domain $\mathbb{R}^n$, the two subspaces associated to A are *Row A* and *Nul A*. Notice that the only vector in their intersection is the zero vector, since

$$\underline{\boldsymbol{x}} \in Row\ A \cap Nul\ A \qquad \Rightarrow \underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{x}} = 0 \qquad \Rightarrow \underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}.$$

So, let

$$\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \ \dots \underline{\boldsymbol{u}}_p\} \qquad \text{be a basis for} \quad Row\ A$$

$$\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ \dots \underline{\boldsymbol{v}}_{n-p}\} \qquad \text{be a basis for } Nul\ A.$$

Then we can check that set of n vectors obtained by taking the union of the two sets,

$$\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \ \dots \underline{\boldsymbol{u}}_p, \underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ \dots \underline{\boldsymbol{v}}_{n-p}\}$$

is actually a basis for $\mathbb{R}^n$. This is because we can show that the n vectors in the set are linearly independent, so they automatically span $\mathbb{R}^n$ and are a basis: To check independence,, let

$$c_1\,\underline{\boldsymbol{u}}_1 + c_2\,\underline{\boldsymbol{u}}_2 + \ \dots + c_p\,\underline{\boldsymbol{u}}_p + d_1\,\underline{\boldsymbol{v}}_1 + d_2\,\underline{\boldsymbol{v}}_2 + \ \dots + d_{n-p}\,\underline{\boldsymbol{v}}_{n-p} = \underline{\boldsymbol{0}}.$$

then

$$c_1\,\underline{\boldsymbol{u}}_1 + c_2\,\underline{\boldsymbol{u}}_2 + \ \dots + c_p\,\underline{\boldsymbol{u}}_p = -d_1\,\underline{\boldsymbol{v}}_1 - d_2\,\underline{\boldsymbol{v}}_2 - \ \dots - d_{n-p}\,\underline{\boldsymbol{v}}_{n-p}.$$

Since the vector on the left is in *Row A* and the one that it equals on the right is in *Nul A*, this vector is the zero vector:

$$c_1\,\underline{\boldsymbol{u}}_1 + c_2\,\underline{\boldsymbol{u}}_2 + \ \dots + c_p\,\underline{\boldsymbol{u}}_p = \underline{0} = -d_1\,\underline{\boldsymbol{v}}_1 - d_2\,\underline{\boldsymbol{v}}_2 - \ \dots - d_{n-p}\,\underline{\boldsymbol{v}}_{n-p}.$$

Since $\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \ \dots \underline{\boldsymbol{u}}_p\}$ and $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ \dots \underline{\boldsymbol{v}}_{n-p}\}$ are linearly independent sets, we deduce from these two equations that

$$c_1 = c_2 = \dots = c_p = 0, \qquad d_1 = d_2 = \dots = d_{n-p} = 0\,.$$

Q.E.D.

So the picture on the previous page is completely general, also for the decomposition of the codomain. One can check that the transformation $T(\underline{x}) = A\,\underline{\boldsymbol{x}}$ restricts to an isomorphism from *Row A* to *Col A*, because it is $1-1$ on these subspaces of equal dimension, so must also be onto. So, T squashes *Nul A*, and maps every translation of *Nul A* to a point in *Col A*. More precisely, Each

$$\underline{\boldsymbol{x}} \in \mathbb{R}^n$$

can be written uniquely as

$$\underline{\boldsymbol{x}} = \underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \qquad \text{with } \underline{\boldsymbol{u}} \in Row\ A,\ \underline{\boldsymbol{v}} \in\ Nul\ A.$$

and

$$T(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = T(\underline{\boldsymbol{u}}) + T(\underline{\boldsymbol{v}}) = T(\underline{u}) \in Col\ (A).$$

As sets,

$$T(\{\underline{\boldsymbol{u}} + Nul\ A\}) = T(\underline{\boldsymbol{u}}).$$