Math 2270-004 Week 9 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 4.6-4.7, 5.1-5.2

Mon Mar 5
- 4.5 General theorems about finite dimensional vector spaces, bases, spanning sets, linearly independent sets and subspaces.

<u>Announcements:</u> whew! (warm-up exercise gone awry :))

<u>Warm-up Exercise</u>: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = -2t_2 - 3t_3$
$x_2 = t_2$
$x_3 = t_3$

$\vec{x} = t_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

Find bases for

Nul A basis: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Row A basis $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

Col A basis $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

Nul A^T basis $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$

$A^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \xrightarrow{\text{reduce}} \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

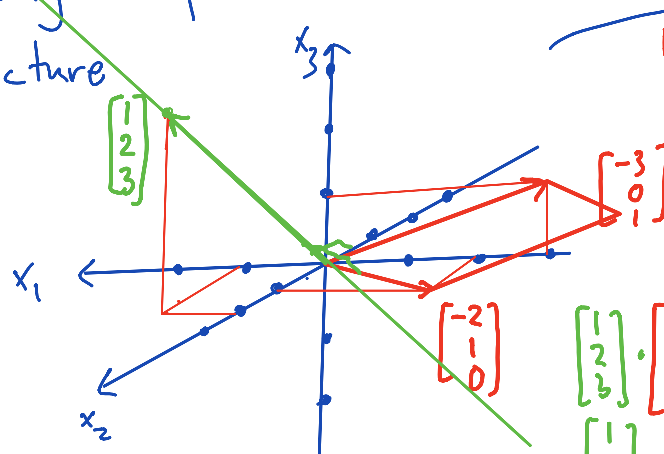
Row A

using dot products, Nul$A \perp$ Row$A$

picture

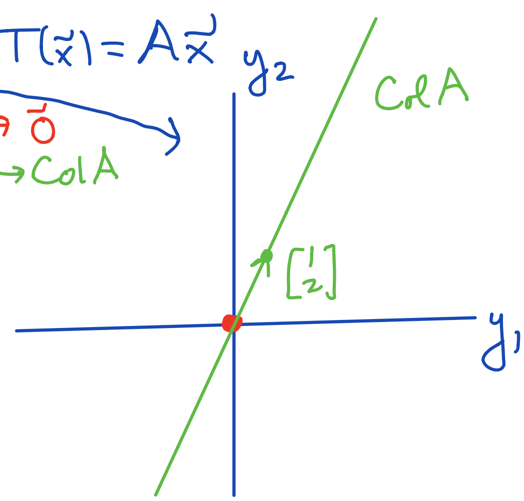
$T(\vec{x}) = A\vec{x}$

Nul$A \mapsto \vec{0}$
Row$A \mapsto$ Col$A$



$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = 0$

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = 0$



Monday Review!

We've been studying *vector spaces*, which are a generalization of $\mathbb{R}^n$. They occur as *subspaces* of $\mathbb{R}^n$; also as vector spaces and subspaces of matrices, and of function spaces.

We've been studying *linear transformations* $T : V \to W$ between vector spaces, which are generalizations of matrix transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ given as $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$.

For an $m \times n$ matrix A there are two interesting subspaces: $Nul\,A = \{\underline{\boldsymbol{x}} \in \mathbb{R}^n : A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}\}$ and $Col\,A = \{\underline{\boldsymbol{b}} \in \mathbb{R}^m : \underline{\boldsymbol{b}} = A\,\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}} \in \mathbb{R}^n\} = span\left\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n\right\}$. (Here we expressed $A = \left[\underline{\boldsymbol{a}}_1\ \underline{\boldsymbol{a}}_2\ \ldots\ \underline{\boldsymbol{a}}_n\right]$ in terms of its columns.) Through homework and food for thought questions we've understood the *rank+nullity Theorem*, that *dim Nul A + dim Col A = n*. This theorem follows from considerations of the reduced row echelon form of A.

In the Friday food for thought questions this past Friday we realized there's a third interesting subspace associated to the matrix A, namely *Row A*, which is the subspace in $\mathbb{R}^n$ spanned by the rows of A. We'll see the fourth and final subspace associated with A tomorrow, and what how these four subspaces are connected to the domain and codomain geometry of the transformation $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$. (This is section 4.6, which we've been secretly thinking about for the past week. We'll utilize many of these ideas again in Chapter 6, e.g. section 6.5.)

We've defined *kernel T* and *range T* for linear transformations $T : V \to W$, generalizing *Nul A* and *Col A* for matrix transformations.

We've defined what it means for a linear transformation $T : V \to W$ to be an *isomorphism,* and checked that in this case the inverse function $T^{-1} : W \to V$ is also a linear transformation (isomorphism) - generalizing the notion of invertible matrix transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ that are given as $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$, with $T^{-1}(\underline{\boldsymbol{y}}) = A^{-1}\,\underline{\boldsymbol{y}}$.

With a basis $\beta = \left\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\right\}$ for a vector space V we can define the coordinate transformation isomorphism $T : V \to \mathbb{R}^n$

$$T\left(c_1\,\underline{\boldsymbol{v}}_1 + c_2\,\underline{\boldsymbol{v}}_2 + \ldots + c_n\,\underline{\boldsymbol{v}}_n\right) := \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = [v]_\beta$$

and use these coordinate systems to answer questions about V.

There is a circle of ideas related to linear independence, span, and basis for vector spaces, which it is good to try and understand carefully. That's what we'll do today. These ideas generalize (and use) ideas we've already explored more concretely, and facts we already know to be true for the vector spaces $\mathbb{R}^n$. (A vector space that does not have a basis with a finite number of elements is said to be *infinite dimensional*. For example the space of all polynomials of arbitrarily high degree is an infinite dimensional vector space. We often study finite dimensional subspaces of infinite dimensional vector spaces.)

$\boxed{\text{Theorem 1}}$ (constructing a basis from a spanning set): Let V be a vector space of dimension at least one, and let $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\} = V$.
Then a subset of the spanning set is a basis for V. (We followed a procedure like this to extract bases for *Col A*.)

proof: If $\{\vec{v}_1, \ldots \vec{v}_p\}$ is not already independent,
reorder the set so that

$$\vec{v}_p = d_1\vec{v}_1 + d_2\vec{v}_2 + \ldots + d_{p-1}\vec{v}_{p-1}$$

(we know one of the vectors is a combo of the others)

so general lin. comb

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_{p-1}\vec{v}_{p-1} + c_p\vec{v}_p$$
$$= c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_{p-1}\vec{v}_{p-1} + c_p\left(d_1\vec{v}_1 + d_2\vec{v}_2 + \ldots + d_{p-1}\vec{v}_{p-1}\right)$$
$$\in span\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_{p-1}\}$$

continue deleting dependent vectors until remaining set is independent, with the same span you began with

* basis *

Theorem 2 Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$. Then any set in V containing more than n elements must be linearly dependent. (We used reduced row echelon form to understand this in $\mathbb{R}^n$.)

Let $\alpha = \{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_N\}$ $\quad N > n$

Show α is dependent:

$$c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2 + \ldots + c_N\vec{\alpha}_N = \vec{0} \qquad \text{solutions?}$$

$$\Longleftrightarrow \left[c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2 + \ldots + c_N\vec{\alpha}_N\right]_\beta = [\vec{0}]_\beta = \vec{0} \in \mathbb{R}^n$$

$$c_1[\vec{\alpha}_1]_\beta + c_2[\vec{\alpha}_2]_\beta + \ldots + c_N[\alpha_N]_\beta = \vec{0}$$

$$n\left\{ \underbrace{\left[[\alpha_1]_\beta \ [\vec{\alpha}_2]_\beta \ \cdots \ [\alpha_N]_\beta\right]}_{N} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = \vec{0} \right.$$

at least $N-n$ free parameters
$\Rightarrow$ lots of dependencies

Theorem 3 Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$. Then no set $\alpha = \{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_p\}$ with $p < n$ vectors can span V. (We know this for $\mathbb{R}^n$.)

proof: Assume α does span. This will lead to a contradiction.

$\Downarrow$

if necessary shrink α by throwing away dependent vectors, to get a basis with $q \le p$ elements

Then we have a basis with fewer than n vectors

Theorem 2 $\Rightarrow \{\vec{b}_1, \vec{b}_2 .. \vec{b}_n\}$ is dependent $\Rightarrow\Leftarrow$.

Theorem 4 Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$. Let $\alpha = \{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_p\}$ be a set of independent vectors that don't span V. Then $p < n$, and additional vectors can be added to the set α to create a basis $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_p, \ldots \underline{\boldsymbol{a}}_n\}$ (We followed a procedure like this when we figured out all the subspaces of $\mathbb{R}^3$.)

proof: $\{\vec{a}_1, .. \vec{a}_p\}$ doesn't span.

pick $\vec{a}_{p+1} \in V$, $\vec{a}_{p+1} \notin \text{span}\{\vec{a}_1, \vec{a}_2, .. \vec{a}_p\}$.

Claim $\{\vec{a}_1, \vec{a}_2, .. \vec{a}_p, \vec{a}_{p+1}\}$ is still independent

pf $c_1\vec{a}_1 + c_2\vec{a}_2 + .. + c_p\vec{a}_p + c_{p+1}\vec{a}_{p+1} = \vec{0}$.

Case 1 $c_{p+1} = 0$

Case 2 $c_{p+1} \neq 0$

to be continued...