Wed Mar 28

- 5.6 Discrete dynamical systems

at least 2 of these are the owl-rat model, with different predation consts

Announcements: § 5.6 HW: 1, 3, 4, 5

two interesting examples of D.D.S.

'til 12:55

Warm-up Exercise: read about the glucose-insulin metabolism model in today's notes, or chat with your neighbor :)

what is a discrete dynamical system, with constant transition matrix?

Example: (See text, page 304). A predator-prey system: "Deep in the redwood forests of California, dusky-footed wood rats povide up to 80 % of the diet for the spotted owl, the main predator of the wood rat..." This model is a simplified version of how the owls and wood rats interact: Denote the owl and wood rat populations at time k months by

$$\vec{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix},$$

1000's of rats.

where O_k is the number of owls in the region studied, and R_k is the number of rats (measured in the thousands). Suppose

$$\begin{cases} O_{k+1} = .5\, O_k + .4\, R_k \\ R_{k+1} = -p\, O_k + 1.1\, R_k \end{cases}$$

where p is a positive parameter (predation constant) to be specified. The $(.5)\, O_k$ in the first equation says that with no wood rats for food, only half the owls will survive each month, while the $1.1\, R_k$ says that with no owls as predators, the rat population will grow by 10 % each month. If the rats are plentiful, the $.4\, R_k$ will tend to make the owl population rise, while the negative term $-p\, O_k$ measures the deaths of rats due to predation by owls. (In fact, $1000\, p$ is the average number of rats eaten by one owl in one month.) Determine the evolution of this system when the predation parameter p is .104.

solution We see that

$$\begin{bmatrix} O_{k+1} \\ R_{k+1} \end{bmatrix} = \begin{bmatrix} .5 & .4 \\ -.104 & 1.1 \end{bmatrix} \begin{bmatrix} O_k \\ R_k \end{bmatrix}.$$

Writing

$$\boldsymbol{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}, \quad A = \begin{bmatrix} .5 & .4 \\ -.104 & 1.1 \end{bmatrix}$$

we see

$$\boxed{\boldsymbol{x}_k = A^k\, \boldsymbol{x}_0}$$

where $\boldsymbol{x}_0 = \begin{bmatrix} O_0 \\ R_0 \end{bmatrix}$ are the owl and rat populations at the start. This is a *dynamical system* because it involves quantities that are changing over time. It is a *discrete* dynamical system because we are letting

with constant transition matrix

time change by discrete amounts (of one month). If we allowed time to vary continuously we would get statements about derivatives and would be studying differential equations instead. (See Math 2280 or 2250.) A is the constant transition matrix.

The way to understand this problem is to use the fact that A is diagonalizable:

Input:

eigenvalues $\begin{pmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{pmatrix}$

Results:

$\lambda_1 \approx 1.02$

$\lambda_2 \approx 0.58$

Corresponding eigenvectors:

$\to$ $v_1 \approx (0.769231, 1)$

$v_2 = (5, 1)$

$E_{\lambda = 1.02} = \text{span}\left\{ \begin{bmatrix} 10 \\ 13 \end{bmatrix} \right\}$

$E_{\lambda = .58} = \text{span}\left\{ \begin{bmatrix} 5 \\ 1 \end{bmatrix} \right\}$.

(An exact eigenvector for $\lambda_1 = 1.02$ is actually $\underline{v}_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix}$.)

<u>Exercise 1</u> Describe the long term behavior of the solutions $\underline{x}_k$ to this problem. Begin by writing $\underline{x}_0$ in terms of the eigenbasis. Then apply A repeatedly. (We could do this in terms of diagonalization and the matrix of A with respect to the eigenbasis, but that would be unneccesarily confusing.)

- $$\underline{x}_0 = c_1 \underline{v}_1 + c_2 \underline{v}_2$$

$$\vec{x}_0 = c_1 \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

$$\vec{x}_1 = A\vec{x}_0 = A\left(c_1 \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix}\right)$$

$$= c_1 A \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_2 A \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\vec{x}_1 = c_1 (1.02) \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_2 (.58) \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\vec{x}_2 = A\vec{x}_1$$

$$(A^2 \vec{x}_0) = c_1 (1.02)^2 \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_2 (.58)^2 \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\vec{x}_k = A^k \vec{x}_0 = c_1 (1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_2 (.58)^k \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

as long $\vec{x}_0 = \begin{bmatrix} O_0 \\ R_0 \end{bmatrix}$ $\underline{\text{not}}$ mult of $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$, as $k \to \infty$ they'll each be increasing 2% a month, with Owl to 1000 rat ration

$\approx$ 10:13

(1300 rats/owl.)

<u>Exercise 2</u> Suppose we have a general discrete dynamical system

$$\underline{\boldsymbol{x}}_k = A^k\, \underline{\boldsymbol{x}}_0$$

and that the matrix A is diagonalizable (over the real numbers, or even over the complex numbers). What can you say about the long term behavior of solutions, depending on the absolute value of the eigenvalues of A?

$A\vec{v}_1 = \lambda_1 \vec{v}_1$

$A\vec{v}_2 = \lambda_2 \vec{v}_2$

$\vdots$

$\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ basis of $\mathbb{R}^n$ $\quad (A_{n\times n})$

or. (basis of $\mathbb{C}^n$)

$$\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$$

$$\vec{x}_1 = A\vec{x}_0 = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \cdots + c_n\lambda_n\vec{v}_n$$

$$\vec{x}_k = A^k\vec{x}_0 = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2 + \ldots + c_n\lambda_n^k\vec{v}_n.$$

(1) If all $|\lambda_i| < 1$ $\quad \lim_{k\to\infty} \vec{x}_k = \vec{0}$ $\quad$ (each $\lambda_j^k \to 0$).

(2) If $\lambda_1 \geq 1$ $\quad$ other $|\lambda_j| < 1$

as $k \to \infty$, the $c_1\lambda_1^k\vec{v}_1$ dominates $\vec{x}_k$

(all the other terms $\to 0$)

- you can study this more systematically

72. Use the method outlined in Exercise 70 to check for which values of the constants a, b, and c the matrix

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix} \text{ is diagonalizable.}$$

73. Prove the Cayley-Hamilton theorem, $f_A(A) = 0$, for diagonalizable matrices A. See Exercise 7.3.54.

74. In both parts of this problem, consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix},$$

with eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$ (see Example 1).

a. Are the column vectors of the matrices $A - \lambda_1 I_2$ and $A - \lambda_2 I_2$ eigenvectors of A? Explain. Does this work for other 2×2 matrices? What about diagonalizable $n \times n$ matrices with two distinct eigenvalues, such as projections or reflections? (*Hint:* Exercise 70 is helpful.)

b. Are the column vectors of

$$A - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

eigenvectors of A? Explain.

"Linear Algebra with Applications"
by Otto Bretscher

7.5 Complex Eigenvalues

complex eigenvalues & evects are important
(book's spotted owl example too.)

Imagine that you are diabetic and have to pay close attention to how your body metabolizes glucose. After you eat a heavy meal, the glucose concentration will reach a peak, and then it will slowly return to the fasting level. Certain hormones help regulate the glucose metabolism, especially the hormone insulin. (Compare with Exercise 7.1.52.) Let $g(t)$ be the excess glucose concentration in your blood, usually measured in milligrams of glucose per 100 milliliters of blood. (*Excess* means that we measure how much the glucose concentration deviates from the fasting level.) A negative value of $g(t)$ indicates that the glucose concentration is below fasting level at time t. Let $h(t)$ be the excess insulin concentration in your blood. Researchers have developed mathematical models for the glucose regulatory system. The following is one such model, in slightly simplified (linearized) form.

$$g_{k+1} = a g_k - b h_k$$
$$h_{k+1} = c g_k + d h_k$$

$$\begin{aligned} g(t+1) &= ag(t) - bh(t) \\ h(t+1) &= cg(t) + dh(t) \end{aligned}$$

(These formulas apply between meals; obviously, the system is disturbed during and right after a meal.)

In these formulas, a, b, c, and d are positive constants; constants a and d will be less than 1. The term $-bh(t)$ expresses the fact that insulin helps your body absorb glucose, and the term $cg(t)$ represents the fact that the glucose in your blood stimulates the pancreas to secrete insulin.

For your system, the equations might be

$$\begin{bmatrix} g_{k+1} \\ h_{k+1} \end{bmatrix} = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix} \begin{bmatrix} g_k \\ h_k \end{bmatrix}$$

$$\begin{aligned} g(t+1) &= 0.9g(t) - 0.4h(t) \\ h(t+1) &= 0.1g(t) + 0.9h(t), \end{aligned}$$

$$\begin{bmatrix} g_0 \\ h_0 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

with initial values $g(0) = 100$ and $h(0) = 0$, after a heavy meal. Here, time t is measured in hours.

After one hour, the values will be $g(1) = 90$ and $h(1) = 10$. Some of the glucose has been absorbed, and the excess glucose has stimulated the pancreas to produce 10 extra units of insulin.

$$\begin{bmatrix} g_1 \\ h_1 \end{bmatrix} = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

The rounded values of $g(t)$ and $h(t)$ in the following table give you some sense for the evolution of this dynamical system.

$$\begin{bmatrix} g_1 \\ \ \end{bmatrix} = \begin{bmatrix} 90 \\ \ \end{bmatrix} \qquad \begin{bmatrix} g_k \\ \ \end{bmatrix} = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix}^k \begin{bmatrix} 100 \\ \ \end{bmatrix}$$

t	0	1	2	3	4	5	6	7	8	15	22	29
$g(t)$	100	90	77	62.1	46.3	30.6	15.7	2.3	−9.3	−29	1.6	9.5
$h(t)$	0	10	18	23.9	27.7	29.6	29.7	28.3	25.7	−2	−8.3	0.3

We can "connect the dots" to sketch a rough trajectory, visualizing the long-term behavior. See Figure 1.

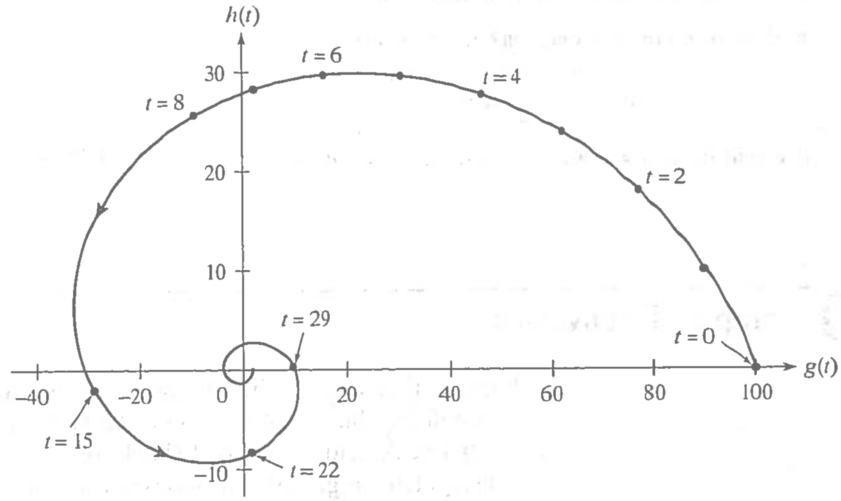

Figure 1

We see that after 7 hours the excess glucose is almost gone, but now there are about 30 units of excess insulin in the system. Since this excess insulin helps to reduce glucose further, the glucose concentration will now fall below fasting level, reaching about −30 after 15 hours. (You will feel awfully hungry by now.) Under normal circumstances, you would have taken another meal in the meantime, of course, but let's consider the case of (voluntary or involuntary) fasting.

We leave it to the reader to explain the concentrations after 22 and 29 hours, in terms of how glucose and insulin concentrations influence each other, according to our model. The *spiraling trajectory* indicates an *oscillatory behavior* of the system: Both glucose and insulin levels will swing back and forth around the fasting level, like a damped pendulum. Both concentrations will approach the fasting level (thus the name).

Another way to visualize this oscillatory behavior is to graph the functions $g(t)$ and $h(t)$ against time, using the values from our table. See Figure 2.

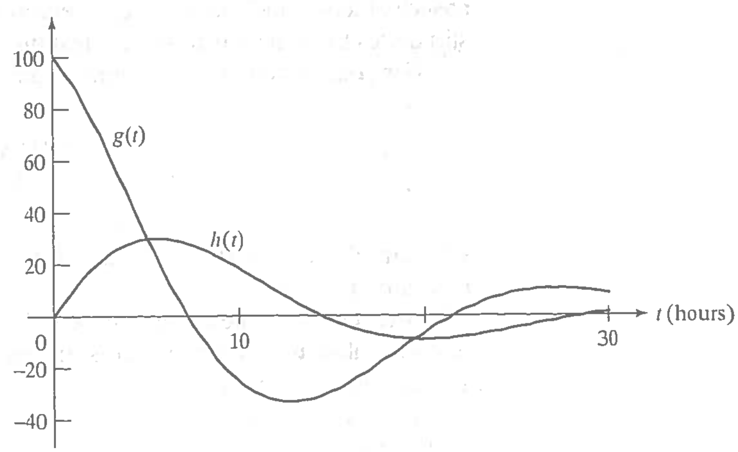

Figure 2

Example: From the algebra yesterday, and after a fair amount of work, For the dynamical system

$$\begin{bmatrix} g_{k+1} \\ h_{k+1} \end{bmatrix} = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix} \begin{bmatrix} g_k \\ h_k \end{bmatrix}$$

and with $\begin{bmatrix} g_0 \\ h_0 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$, one can calculate and understand the spiral picture...

$$\begin{bmatrix} g_k \\ h_k \end{bmatrix} = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix}^k \begin{bmatrix} 100 \\ 0 \end{bmatrix} = = .92^k \begin{bmatrix} 100\cos(k\theta) \\ 50\sin(k\theta) \end{bmatrix}$$

$\theta \approx .22$ radians.

lie on ellipse

$$\frac{g^2}{100^2} + \frac{h^2}{50^2} = 1$$

yipes!

For $r = \sqrt{.85} \approx .92$, $\theta = \arctan\left(\frac{2}{9}\right) \approx .22$ radians.

$$B = r\,P\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}P^{-1}$$

$$B^2 = r^2\,P\begin{bmatrix} \cos(2\,\theta) & -\sin(2\,\theta) \\ \sin(2\,\theta) & \cos(2\,\theta) \end{bmatrix}P^{-1}$$

$$B^n = r^n\,P\begin{bmatrix} \cos(n\,\theta) & -\sin(n\,\theta) \\ \sin(n\,\theta) & \cos(n\,\theta) \end{bmatrix}P^{-1}$$

$$B^n\begin{bmatrix} 100 \\ 0 \end{bmatrix} = .92^n\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \cos(n\,\theta) & -\sin(n\,\theta) \\ \sin(n\,\theta) & \cos(n\,\theta) \end{bmatrix}\frac{1}{2}\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

$$.92^n\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \cos(n\,\theta) & -\sin(n\,\theta) \\ \sin(n\,\theta) & \cos(n\,\theta) \end{bmatrix}\begin{bmatrix} 0 \\ -50 \end{bmatrix}$$

$$= .92^n\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 50\sin(n\theta) \\ -50\cos(n\theta) \end{bmatrix}$$

$$= .92^n\begin{bmatrix} 100\cos(n\theta) \\ 50\sin(n\theta) \end{bmatrix}$$