Tues Mar 27

- 5.5 Complex eigenvalues and eigenvectors

Announcements:

- 5.5 HW: 1, 7, 11, 13
- Quiz tomorrow: find eigenvals & eigenvcts for a 2x2 matrix (they will be complex)
- Wed, Fri: a task of applications
- Monday starts Chpts 6, 7 “orthogonality” & Spectral theorem - big, important unit.

Warm-up Exercise:

Find the eigenvalues of

\[
A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}
\]
\[
\lambda = 1 \pm i
\]

\[
\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = (\lambda-1)^2 + 1 = 0
\]

\[
(\lambda-1)^2 = -1
\]

\[
\lambda - 1 = \pm i
\]

\[
\lambda = 1 \pm i
\]
We'll focus on $2 \times 2$ matrices, for simplicity. In this case it will turn out that a matrix with real entries and complex eigenvalues is always similar to a rotation-dilation matrix.

**Definition** A matrix of the form $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is called a *rotation-dilation* matrix, because for 

$$r = \sqrt{a^2 + b^2}$$

we can rewrite $A$ as

$$A = r \begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = r \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$ 

So the transformation $T(x) = A x$ rotates vectors by an angle $\theta$ and then scales them by a factor of $r$. (So $A^2$ rotates by an angle $2 \theta$ and scales by $r^2$; $A^3$ rotates by an angle $3 \theta$ and scales by $r^3$, etc.

**Exercise 1** Draw the transformation picture for 

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and interpret this transformation as a rotation-dilation.
Exercise 2) What are the eigenvalues of a rotation-dilation matrix $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$?

$$\begin{vmatrix} A - \lambda I \end{vmatrix} = \begin{vmatrix} a - \lambda & -b \\ b & a - \lambda \end{vmatrix} = (\lambda - a)^2 + b^2 = 0$$

$$\begin{align*}
(\lambda - a)^2 &= -b^2 \\
\lambda - a &= \pm ib \\
\lambda &= a \pm ib
\end{align*}$$

It is possible for a matrix $A$ with real entries to be diagonalizable if one allows complex scalars and vectors, even if it's not diagonalizable with real eigenvalues and eigenvectors. You saw an example of that on a food for thought problem, if you weren't afraid. We'll use a matrix today that we'll use later as well, in section 5.6, to study an interesting discrete dynamical system. This matrix is not a rotation-dilation matrix, but it is similar to one, and that fact will help us understand the discrete dynamical system.

Exercise 3) Let

$$B = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix}$$

Find the (complex) eigenvalues and eigenvectors for $B$.

$$\begin{vmatrix} .9 - \lambda & -.4 \\ .1 & .9 - \lambda \end{vmatrix} = (\lambda - .9)^2 + .04 = 0$$

$$\begin{align*}
(\lambda - .9)^2 &= -.04 \\
\lambda - .9 &= \pm .2i \\
\lambda &= .9 \pm .2i
\end{align*}$$

you can do all of linear algebra using complex scalars, and $\mathbb{C}$ instead of $\mathbb{R}^n$.

in general: expand & use quad. formula

in complex, these rows are multiples of each other!!

$$E_{\lambda = .9 + .2i} = \begin{bmatrix} -2i & -.4 \\ 1 & -2i \end{bmatrix} = 0$$

$$V^T = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

$$E_{\lambda = .9 + .2i} = \text{span} \left\{ \begin{bmatrix} 2i \\ 1 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} -2i \\ -1 \end{bmatrix} \right\}$$

all complex multiple
\[
E_{\lambda = 0.9 - 2i} \begin{bmatrix} 0.2i & -0.4 \\ 0.1 & 0.2i \end{bmatrix} E_{\lambda = 0.9 - 2i} = \text{span}\{ \begin{bmatrix} -2i \\ 1 \end{bmatrix} \}
\]

For conjugate eigenvalues, the eigenvector was the conjugate of the 1st one, replace all "i"'s with "-i"'s.
General facts we saw illustrated in the example, about complex eigenvalues and eigenvectors: Let $A$ be a matrix with real entries, and let

$$A \mathbf{v} = \lambda \mathbf{v}$$

with $\lambda = a + b i$, $\mathbf{v} = \mathbf{u} + i \mathbf{w}$ complex, $(a, b \in \mathbb{R}, \mathbf{u}, \mathbf{w} \in \mathbb{R}^n)$. Then we write

$$\text{Re} \lambda = a, \quad \text{Im} \lambda = b$$

$$\text{Re} \mathbf{v} = \mathbf{u}, \quad \text{Im} \mathbf{v} = \mathbf{w}.$$

So, the equation $A \mathbf{v} = \lambda \mathbf{v}$ expands as

$$A (\mathbf{u} + i \mathbf{w}) = (a + b i)(\mathbf{u} + i \mathbf{w}).$$

It will always be true then that the conjugate $\lambda = a - b i$ is also an eigenvalue, and the conjugate vector $\mathbf{v} = \mathbf{u} - i \mathbf{w}$ will be a corresponding eigenvector, because it will satisfy

$$A (\mathbf{u} - i \mathbf{w}) = (a - b i)(\mathbf{u} - i \mathbf{w}).$$

Exercise 4 Verify that if the first eigenvector equation holds, then

$$A \mathbf{u} = a \mathbf{u} - b \mathbf{w}$$
$$A \mathbf{w} = b \mathbf{u} + a \mathbf{w}$$

Then check that these equalities automatically make the second conjugate eigenvector equation true as well.
Theorem: Let $A$ be a real $2 \times 2$ matrix with complex eigenvalues. Then $A$ is similar to a rotation-dilation matrix.

Proof: Let a complex eigenvalue and eigenvector be given by $\lambda = a + b i, v = u + i w$ complex, ($a, b \in \mathbb{R}, u, w \in \mathbb{R}^n$) Choose $P = \begin{bmatrix} \text{Re} v & \text{Im} v \end{bmatrix} = \begin{bmatrix} u & w \end{bmatrix}$

(One can check that $\{u, w\}$ is automatically independent.) Then, using the equations of Exercise 4, we mimic what we did for diagonalizable matrices...

$$A P = P \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

(The matrix on the right is a rotation-dilation matrix ... nobody ever said what the sign of $b$ was. :-))
It's a mess, but we can carry out the procedure of the theorem, for the matrix $B$ in exercise 3,

$$B = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix}$$

using $\lambda = .9 - .2i$, $v = \begin{bmatrix} -2i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, one gets

$$P = [ \text{Re} \ v \ \ \text{Im} \ v] = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \quad P^{-1} = \frac{1}{2} \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$$

$$P^{-1}BP = \frac{1}{2} \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$$

$$P^{-1}BP = \begin{bmatrix} .9 & -.2 \\ .2 & .9 \end{bmatrix} = \sqrt{.85} \begin{bmatrix} .9 \\ \sqrt{.85} & \sqrt{.85} \\ \frac{.2}{\sqrt{.85}} & \frac{.9}{\sqrt{.85}} \end{bmatrix}$$

$$P^{-1}BP = r \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$  

for $r = \sqrt{.85} \approx .92$, $\theta = \arctan \left( \frac{2}{9} \right) \approx .22$ radians.