Tues Mar 27

- 5.5 Complex eigenvalues and eigenvectors

Announcements:
- §5.5 HW: 1, 7, 11, 13 •
- quiz tomorrow: find eigenvals & eigenvects for a 2×2 matrix (they will be complex)
- Wed, Fri: a taste of applications
- Monday starts Chptrs 6, 7 "orthogonality" & Spectral theorem big, important unit.

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Warm-up Exercise: Find the eigenvalues of

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \lambda = 1 \pm i$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = (\lambda-1)^2 + 1 = 0$$

$$(\lambda-1)^2 = -1$$

$$\lambda - 1 = \pm i$$

$$\boxed{\lambda = 1 \pm i}$$

We'll focus on 2 × 2 matrices, for simplicity. In this case it will turn out that a matrix with real entries and complex eigenvalues is always similar to a rotation-dilation matrix.

Definition A matrix of the form $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is called a *rotation-dilation* matrix, because for $r = \sqrt{a^2+b^2}$ we can rewrite A as

$$A = r\begin{bmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{bmatrix} = r\underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\text{rotates by angle } \theta}.$$

$$\frac{a}{r} = \frac{a}{\sqrt{a^2+b^2}} \qquad \frac{b}{r} = \frac{b}{\sqrt{a^2+b^2}}$$

So the transformation $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$ rotates vectors by an angle θ and then scales them by a factor of r. (So A^2 rotates by an angle 2 θ and scales by r^2; A^3 rotates by an angle 3 θ and scales by r^3, etc.

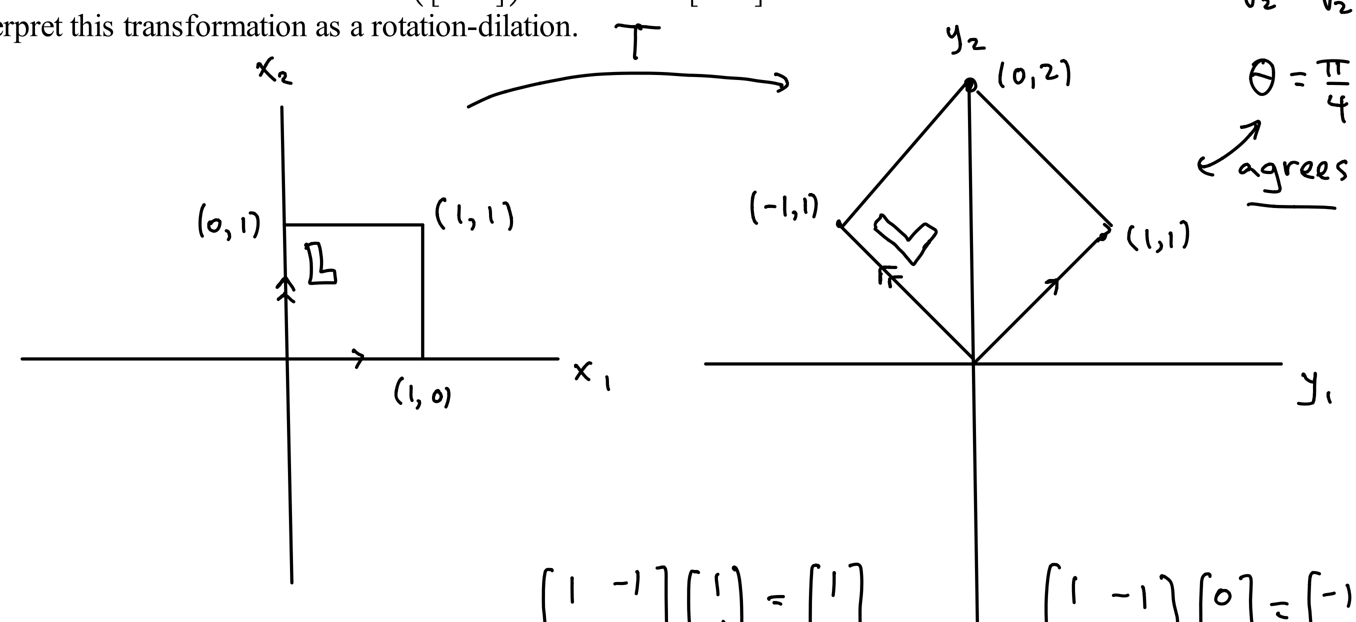
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Exercise 1 Draw the transformation picture for

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \sqrt{2}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and interpret this transformation as a rotation-dilation.



$\theta = \frac{\pi}{4}$ rad

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Exercise 2) What are the eigenvalues of a rotation-dilation matrix $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$?

$$|A - \lambda I| = \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = (\lambda - a)^2 + b^2 = 0$$

$$(\lambda - a)^2 = -b^2$$

$$\lambda - a = \pm ib$$

$$\lambda = a \pm ib$$

It is possible for a matrix A with real entries to be diagonalizable if one allows complex scalars and vectors, even if it's not diagonalizable with real eigenvalues and eigenvectors. You saw an example of that on a food for thought problem, if you weren't afraid. We'll use a matrix today that we'll use later as well, in section 5.6, to study an interesting discrete dynamical system. This matrix is not a rotation-dilation matrix, but it is *similar* to one, and that fact will help us understand the discrete dynamical system.

Exercise 3) Let

$$B = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix}$$

Find the (complex) eigenvalues and eigenvectors for B.

you can do all of linear algebra using complex scalars, and $\mathbb{C}^n$ instead of $\mathbb{R}^n$

$$\begin{vmatrix} .9-\lambda & -.4 \\ .1 & .9-\lambda \end{vmatrix} = (\lambda - .9)^2 + .04 = 0$$

in general: expand & use quad. formula

$$(\lambda - .9)^2 = -.04$$

$$\lambda - .9 = \pm .2i$$

$$\lambda = .9 \pm .2i$$

$E_{\lambda = .9 + .2i}$

$Nul(B - \lambda I)$

$$\left[\begin{array}{cc|c} -.2i & -.4 & 0 \\ .1 & -.2i & 0 \end{array}\right]$$

these rows are complex multiples of each other!!

$10R_2 \rightarrow R_1$

$10R_1 \rightarrow R_2$

$$\left[\begin{array}{cc|c} 1 & -2i & 0 \\ -2i & -4 & 0 \end{array}\right]$$

$2iR_1 + R_2 \rightarrow R_2$

$$\left[\begin{array}{cc|c} 1 & -2i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$2i(-2i) - 4 = 4 - 4 = 0$

$$\vec{v} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

$$E_{\lambda = .9 + .2i} = span\left\{ \begin{bmatrix} 2i \\ 1 \end{bmatrix} \right\} \quad \left(= span\left\{ \begin{bmatrix} -2 \\ i \end{bmatrix} \right\} \right)$$

all complex mults

$E_{\lambda=.9-.2i}$

$$\left[\begin{array}{cc|c} .2i & -.4 & 0 \\ .1 & .2i & 0 \end{array}\right]$$

$10R_2 \to R_1$, $-10R_1 \to R_2$

$$\left[\begin{array}{cc|c} 1 & 2i & 0 \\ 2i & -4 & 0 \end{array}\right]$$

$-2iR_1 + R_2 \to R_2$

$$\left[\begin{array}{cc|c} 1 & 2i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\tilde{v} = \begin{bmatrix} -2i \\ 1 \end{bmatrix} \qquad E_{\lambda=.9-.2i} = \text{span}\left\{ \begin{bmatrix} -2i \\ 1 \end{bmatrix} \right\}$$

for $\underbrace{\text{conjugate}}$ eigenvalue, the eigenvector was the conjugate of the 1st one

replace all "i"'s with "$-i$"'s.

General facts we saw illustrated in the example, about complex eigenvalues and eigenvectors: Let A be a matrix with real entries, and let

$A_{n\times n}$

$$A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$$

with $\lambda = a + b\,i$, $\underline{\boldsymbol{v}} = \underline{\boldsymbol{u}} + i\,\underline{\boldsymbol{w}}$ complex, $(a, b \in \mathbb{R}, \underline{\boldsymbol{u}}, \underline{\boldsymbol{w}} \in \mathbb{R}^n)$. Then we write

$$\operatorname{Re}\lambda = a, \quad \operatorname{Im}\lambda = b$$

$$\operatorname{Re}\underline{\boldsymbol{v}} = \underline{\boldsymbol{u}}, \qquad \operatorname{Im}\underline{\boldsymbol{v}} = \underline{\boldsymbol{w}}.$$

So, the equation $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$ expands as

$$A\,(\underline{\boldsymbol{u}} + i\,\underline{\boldsymbol{w}}) = (a + b\,i)(\underline{\boldsymbol{u}} + i\,\underline{\boldsymbol{w}}).$$

$$A\vec{u} + iA\vec{w} = (a\vec{u} - b\vec{w}) + i(b\vec{u} + a\vec{w})$$

It will always be true then that the conjugate $\bar{\lambda} = a - b\,i$ is also an eigenvalue, and the conjugate vector $\bar{\underline{\boldsymbol{v}}} = \underline{\boldsymbol{u}} - i\,\underline{\boldsymbol{w}}$ will be a corresponding eigenvector, because it will satisfy

$$A\,(\underline{\boldsymbol{u}} - i\,\underline{\boldsymbol{w}}) = (a - b\,i)(\underline{\boldsymbol{u}} - i\,\underline{\boldsymbol{w}})$$

$$A\vec{u} - iA\vec{w} = (a\vec{u} - b\vec{w}) + i(-b\vec{u} - a\vec{w})$$

Exercise 4 Verify that if the first eigenvector equation holds, then

$$A\,\underline{\boldsymbol{u}} = a\,\underline{\boldsymbol{u}} - b\,\underline{\boldsymbol{w}}$$
$$A\,\underline{\boldsymbol{w}} = b\,\underline{\boldsymbol{u}} + a\,\underline{\boldsymbol{w}}$$

Then check that these equalities automatically make the second conjugate eigenvector equation true as well.

$\vec{v} = \vec{u} + i\vec{w}$
eigenvector
$\lambda = a + ib$
eigenvalue

$A\vec{u} = a\vec{u} - b\vec{w}$
$A\vec{w} = b\vec{u} + a\vec{w}$

Theorem Let A be a real 2×2 matrix with complex eigenvalues. Then A is similar to a rotation-dilation matrix.

proof: Let a complex eigenvalue and eigenvector be given by $\lambda = a + b\,i$, $\underline{\boldsymbol{v}} = \underline{\boldsymbol{u}} + i\,\underline{\boldsymbol{w}}$ complex, ($a, b \in \mathbb{R}$, $\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}} \in \mathbb{R}^n$) Choose

$$P = [\ \mathrm{Re}\,\underline{\boldsymbol{v}} \quad \mathrm{Im}\,\underline{\boldsymbol{v}}\,] = [\,\underline{\boldsymbol{u}}\ \ \underline{\boldsymbol{w}}\,]$$

(One can check that $\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}}\}$ is automatically independent.) Then, using the equations of Exercise 4, we mimic what we did for diagonalizable matrices...

$$A\,\underbrace{[\,\underline{\boldsymbol{u}}\ \ \underline{\boldsymbol{w}}\,]}_{P} = [\,\overbrace{a\,\underline{\boldsymbol{u}} - b\,\underline{\boldsymbol{w}}}^{A\vec{u}},\ \overbrace{b\,\underline{\boldsymbol{u}} + a\,\underline{\boldsymbol{w}}}^{A\vec{w}}\,]$$

$$= [\,\underline{\boldsymbol{u}}\ \ \underline{\boldsymbol{w}}\,]\begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

$$A\,P = P\underbrace{\begin{bmatrix} a & b \\ -b & a \end{bmatrix}}_{\text{rotation-dilation matrix}}$$

$$P^{-1}\,A\,P = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

(The matrix on the right is a rotation-dilation matrix ... nobody ever said what the sign of b was. :-))

It's a mess, but we can carry out the procedure of the theorem, for the matrix B in exercise 3,

$$B = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix}$$

using $\lambda = .9 - .2\,i$, $\underline{\mathbf{v}} = \begin{bmatrix} -2\,i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, one gets

$$P = [\ \mathrm{Re}\,\underline{\mathbf{v}} \quad \mathrm{Im}\,\underline{\mathbf{v}}] = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \qquad P^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$$

$$P^{-1} B\, P = \frac{1}{2}\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$$

$$P^{-1} B\, P = \begin{bmatrix} .9 & -.2 \\ .2 & .9 \end{bmatrix} = \sqrt{.85}\begin{bmatrix} \frac{.9}{\sqrt{.85}} & -\frac{.2}{\sqrt{.85}} \\ \frac{.2}{\sqrt{.85}} & \frac{.9}{\sqrt{.85}} \end{bmatrix}$$

$\sqrt{.9^2 + .2^2} = \sqrt{.85}$

$$P^{-1} B\, P = r\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

for $r = \sqrt{.85} \approx .92$, $\theta = \arctan\left(\frac{2}{9}\right) \approx .22$ radians.