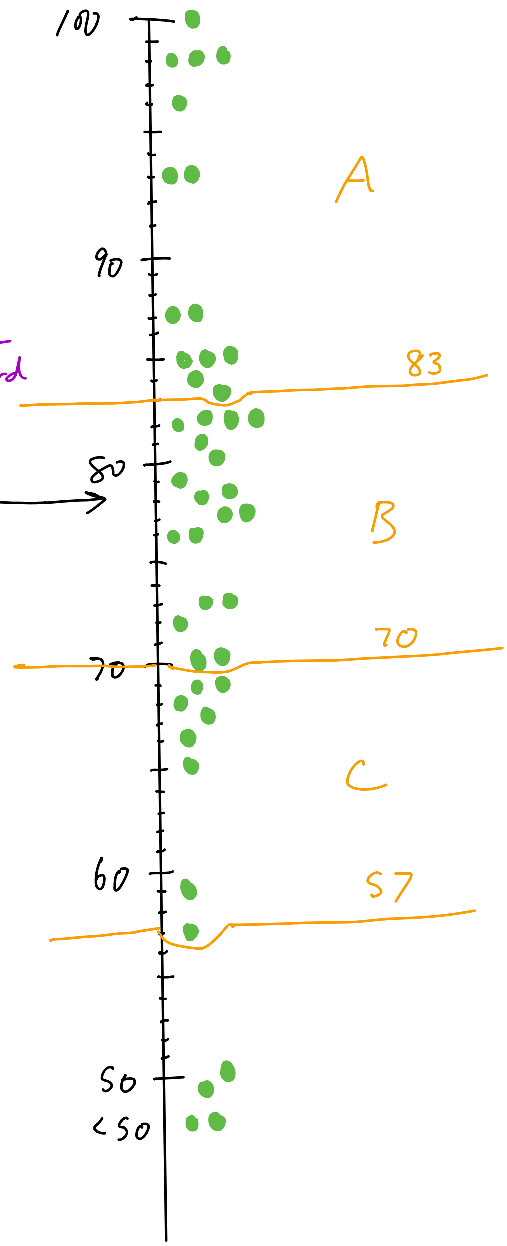
2270-004
Exam 2 scores n=44

no warm-up problem:
look over exams & soltns

• eigenvalue-eigenvectr problem was hard
• TF is always tricky.

median



Math 2270-004 Week 11 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 5.4-5.6

Mon Mar 26
- 5.4 matrices for linear transformations as a general framework to understand change of bases, diagonalization, and similar matrices.

Announcements: §5.4 HW: 1, 3, 5, 11, 13, 17
§5.4 consolidates a lot of the ideas we've been discussing.

Warm-up Exercise: look over exams

Monday Review and look ahead:

We've been studying *linear transformations* $T : V \rightarrow W$ between vector spaces, which include matrix transformations $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ given as $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$.

We've been studying how coordinates change when we change bases in $\mathbb{R}^n$.

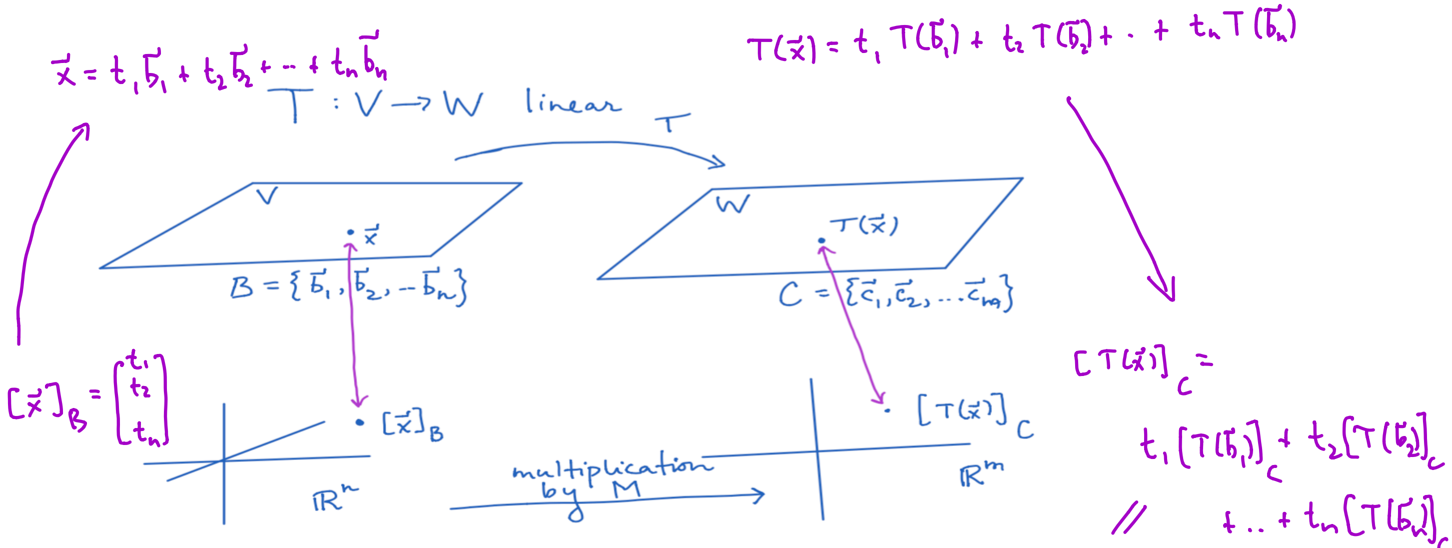
The last thing we studied in depth before the midterm was eigenvectors and eigenvalues for square matrices A, and the notion of diagonalizability, which we understood in an algebraic sense.

On the Wednesday before the midterm we introduced section 5.4, about how linear transformations $T : V \rightarrow W$ are associated with matrix transformations from $\mathbb{R}^n \rightarrow \mathbb{R}^m$, once we choose bases for V and W. We didn't have time to explain how this general framework is connected to all of our previous change of coordinates discussion, to matrix diagonalizability, and to the more general notion of similar matrices. That's what we'll do today.

Tomorrow we'll study section 5.5 on complex eigenvalues and eigenvectors. To understand the geometry of matrix transformations with complex eigendata we'll use "similar matrices" notions from today, to see that (in the 2×2 case), such matrices are similar to "rotation-dilations". You saw a hint of this on a food for thought problem before break, if you dared.

Wednesday we'll start section 5.6 on discrete dynamical systems, and we'll continue that discussion into Friday with google page rank. These section 5.6 topics are more expository than comprehensive, and for fun. There will be some follow-up homework problems.

Recall, if we have a linear transformation $T : V \to W$ and bases $B = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$ in V, $C = \{\underline{\boldsymbol{c}}_1, \underline{\boldsymbol{c}}_2, \ldots \underline{\boldsymbol{c}}_m\}$ in W, then the matrix of T with respect to these two bases transforms the B coordinates of vectors $\underline{\boldsymbol{x}} \in V$ to the C coordinates of $T(\underline{\boldsymbol{x}})$ in a straightforward way:



$$M = \Big[[T(\vec{b}_1)]_C \;\; [T(\vec{b}_2)]_C \;\; \cdots \;\; [T(\vec{b}_n)]_C \Big] \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}$$

<u>Exercise 1)</u> Explain why the columns of the matrix M *have to be* the C coordinate vectors of T applied the B basis vectors. Do this two ways: (1) using the chart. AND (2) seeing what must happen when you multiply M by the standard basis vectors. This should help you remember in case you get confused.

another way to remember M: takes coords in domain wrt B

$M[\vec{x}]_B = [T(\vec{x})]_C$

and mult by M, get coords for $T(\vec{x})$ in codomain.

So, $M\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [T(\vec{b}_1)]_C$ = 1st col. of M.

$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [\vec{b}_1]_B$

$M\vec{e}_j = [T(\vec{b}_j)]_C$ = col$_j$ M.

$\vec{e}_j = [\vec{b}_j]_B$

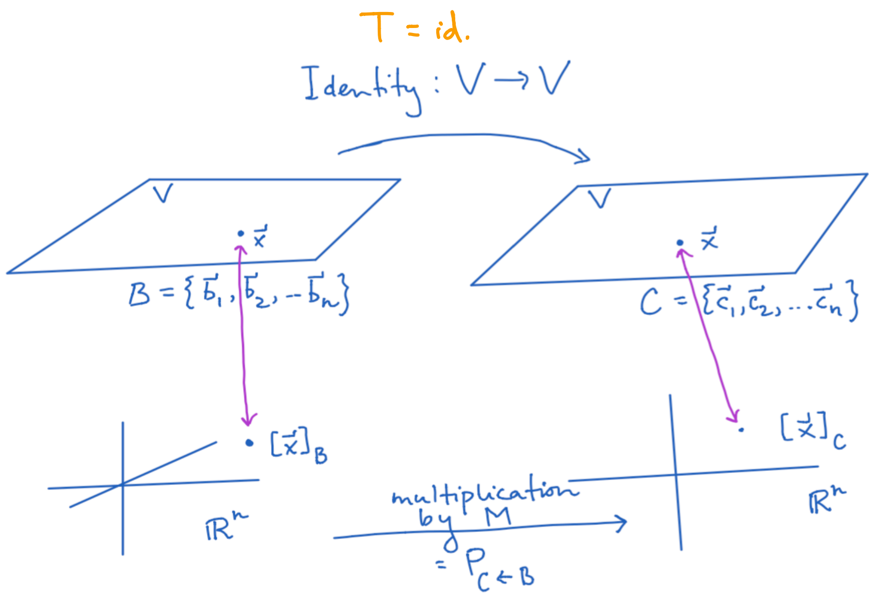
<u>Expl</u>: $\{\vec{b}_1, \vec{b}_2\}$ basis for V $\quad \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ basis for W

$$T(\vec{b}_1) = 3\vec{c}_1 - \vec{c}_2 + \vec{c}_3$$
$$T(\vec{b}_2) = -2\vec{c}_2 + 3\vec{c}_3$$

find M

So $M = \begin{bmatrix} 3 & 0 \\ -1 & -2 \\ 1 & 3 \end{bmatrix}$

Exercise 2) Fill in the matrix M for changing coordinates in a general vector space. We focused on changing coordinates in $\mathbb{R}^n$ in section 4.7, which is a special case of this when $V = \mathbb{R}^n$ itself. The text also discussed the more general context below, in that section.



$$* \quad \left[[T(\vec b_1)]_C \ \ [T(\vec b_2)]_C \ \cdots \ [T(\vec b_n)]_C \right]$$

$$M = \left[[\vec b_1]_C \ \ [\vec b_2]_C \ \cdots \ [\vec b_n]_C \right]$$

or $\quad M \underset{\substack{\uparrow \\ [\vec b_j]_B}}{\vec e_j} = [\vec b_j]_C$

$\qquad\qquad = \mathrm{col}_j(M)$

Example: On the midterm: Let $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $C = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ be two bases for $V = \mathbb{R}^2$.

Find the change of coordinates matrix $P_{C \leftarrow B}$:

Solution:

$$P_{C \leftarrow B} = \left[[\boldsymbol{b}_1]_C \ [\boldsymbol{b}_2]_C \right]$$

Since $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & 1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ the answer was $P_{C \leftarrow B} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$.

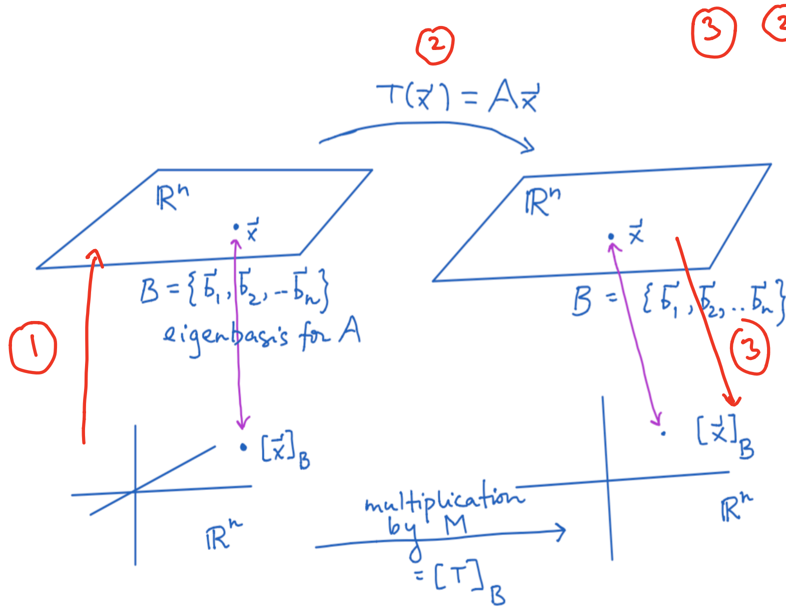
(Which, as the diagram indicates, could also have been computed as

$$\left[\quad P_{C \leftarrow B} = \underbrace{P_{C \leftarrow E}}_{2^{nd}} \underbrace{P_{E \leftarrow B}}_{1^{st}} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} .) \quad \right]$$

Exercise 4) What if a matrix A is diagonalizable? What is the matrix of $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$ with respect to the eigenbasis? How does this connect to our matrix identities for diagonalization? Fill in the matrix M below, and then compute another way to express it, as a triple product using the diagram.

$$M = D = \underset{③}{P_{B\leftarrow E}}\ \underset{②}{A}\ \underset{①}{P_{E\leftarrow B}} = P^{-1}AP$$

evects in cols



$A\vec{b}_1 = \lambda_1\vec{b}_1$

$* \ A\vec{b}_2 = \lambda_2\vec{b}_2$

etc.

$$M = \left[\, [T(\vec{b}_1)]_B \ \ [T(\vec{b}_2)]_B \ \cdots \ [T(\vec{b}_n)]_B \,\right]$$

$$= \underset{\quad [A\vec{b}_1]_B \qquad [A\vec{b}_2]_B}{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & 0 & & \vdots \\ 0 & \vdots & & \\ & 0 & & \lambda_n \end{bmatrix}} = D$$

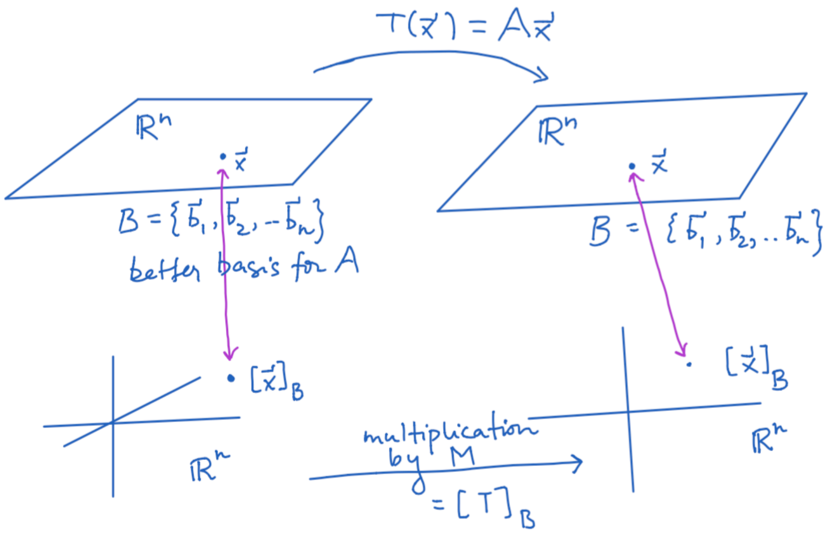
$A\vec{b}_1 = \lambda_1\vec{b}_1 + 0\vec{b}_2 + 0\vec{b}_3 + \cdots + 0\vec{b}_n$

Example, from the week before break:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \qquad E_{\lambda=4} = span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\} \qquad E_{\lambda=1} = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} \qquad B = \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

Write the various matrices corresponding to the diagram above.

Even if the matrix A is not diagonalizable, there may be a better basis to help understand the transformation $T(\mathbf{x}) = A\,\mathbf{x}$. The diagram on the previous page didn't require that B be a basis of eigenvectors....maybe it was just a "better" basis than the standard basis, to understand T.



Exercise 5 (If we have time - this one is not essential.) Try to pick a better basis to understand the matrix transformation $T(\mathbf{x}) = C\,\mathbf{x}$, even though the matrix C is not diagonalizable. Compute $M = P^{-1}A\,P$ or compute M directly, to see if it really is a "better" matrix.

$$C = \begin{bmatrix} 4 & 4 \\ -1 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 4-\lambda & 4 \\ -1 & 0-\lambda \end{vmatrix} = (4-\lambda)(-\lambda) + 4 = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$$

$E_{\lambda=2}$

$\mathrm{Nul}\,(C-2I)$

$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ -1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

basis: $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$

$$B = \left\{ \underset{\vec{b}_1}{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}, \underset{\vec{b}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\}.$$

$$T(\vec{x}) = C\vec{x}.$$

$$[T]_B = \Big[[C\vec{b}_1]_B \quad [C\vec{b}_2]_B \Big]$$

$$= \begin{bmatrix} 2 & ? \\ 0 & \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$$

$$C\vec{b}_1 = 2\vec{b}_1$$

$$C\vec{b}_2 = \begin{bmatrix} 4 & 4 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

coords

$$\left[\begin{array}{cc|c} -2 & 0 & 4 \\ 1 & 1 & 0 \end{array}\right]$$

$$= -2\begin{bmatrix} -2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[C\vec{b}_2]_B = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

OR $\quad [T]_B = \underset{B \leftarrow E}{P} \, C \, \underset{E \leftarrow B}{P} = P^{-1} C P$

$$\begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 & 4 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}$$

would get same ans!