Fri Mar 2

- 4.6 The four subspaces associated with a matrix. the rank of a matrix.

Announcements: Finish Tuesday notes

I think we'll finish Wed, but if we don't, that's fine

- fft

Warm-up Exercise: See Tuesday notes

From Tuesday notes, finished on $\boxed{\text{Friday}}$

Theorem Let V, W be vector spaces, and $T : V \rightarrow W$ a linear transformation. If T is $1-1$ and onto, then the inverse function T^{-1} is also a linear transformation, $T^{-1} : W \rightarrow V$. In this case, we call T an *isomorphism*.

proof: We have to check that for all $\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}} \in W$ and all $c \in \mathbb{R}$,

check

$$T^{-1}(\underline{\boldsymbol{u}} + \underline{\boldsymbol{w}}) = T^{-1}(\underline{\boldsymbol{u}}) + T^{-1}(\underline{\boldsymbol{w}})$$

$$T^{-1}(c\,\underline{\boldsymbol{u}}) = c\,T^{-1}(\underline{\boldsymbol{u}}).$$

since T is 1-1
suffices to check that T of LHS's $=$ T RHS's

T of RHS $T\left(T^{-1}(\vec{u}) + T^{-1}(\vec{w})\right) \overset{T \text{ linear}}{=} T(T^{-1}(\vec{u})) + T(T^{-1}(\vec{w}))$
$= \vec{u} + \vec{w}$

T of LHS. : $T(T^{-1}(\vec{u}+\vec{v}))$
$= \vec{u} + \vec{v}$

for scalar multiplication : $T(T^{-1}(c\vec{u})) = c\vec{u}$, $T(cT^{-1}(\vec{u})) \overset{T \text{ is linear}}{=} c\,TT^{-1}(\vec{u})$
$= c\vec{u}$

Theorem Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$. Then the coordinate transform $T : V \rightarrow \mathbb{R}^n$ defined by

$$\boxed{T(\underline{\boldsymbol{v}}) = [\underline{\boldsymbol{v}}]_\beta}$$

is linear, and it is an isomorphism. Friday!!

let $\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n$ so $[\vec{v}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$\vec{w} = d_1\vec{b}_1 + d_2\vec{b}_2 + \cdots + d_n\vec{b}_n$

$\vec{v}+\vec{w} = (c_1+d_1)\vec{b}_1 + (c_2+d_2)\vec{b}_2 + \cdots + (c_n+d_n)\vec{b}_n$ $\quad [\vec{w}]_\beta = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$

(1). $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$ $\rightarrow$ $[\vec{v}+\vec{w}]_\beta = \begin{bmatrix} c_1+d_1 \\ c_2+d_2 \\ \vdots \\ c_n+d_n \end{bmatrix} = [\vec{v}]_\beta + [\vec{w}]_\beta$

$c\vec{v} = c(c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n)$
$= cc_1\vec{b}_1 + cc_2\vec{b}_2 + \cdots$

(2) $T(c\vec{v}) = cT(\vec{v})$ $\quad [c\vec{v}]_\beta = \begin{bmatrix} cc_1 \\ cc_2 \\ \vdots \\ cc_n \end{bmatrix} = c[\vec{v}]_\beta$

I know $T^{-1} : \mathbb{R}^n \rightarrow V$

$T^{-1}\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = e_1\vec{v}_1 + e_2\vec{v}_2 + \cdots + e_n\vec{v}_n.$

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Friday warm-up (Exercise 2 in Tuesday notes)

<u>Exercise 2</u>: Use coordinates with respect to the basis $\underbrace{\{1, t, t^2\}}_{=\beta}$, to check whether or not the set of polynomials $\{p_1(t), p_2(t), p_3(t)\}$ is a basis for $\boldsymbol{P}_2$, where

$$p_1(t) = 1 + t^2 \qquad [p_1]_\beta = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ etc.}$$
$$p_2(t) = 2 + 3t + t^2$$
$$p_3(t) = -3t + t^2 .$$

i.e. check whether the coord vectors in $\mathbb{R}^3$ are independent & span $\mathbb{R}^3$. Then make conclusions about $\{p_1, p_2, p_3\}$

$$p_2 = 2p_1 - p_3$$
$$2p_1 - p_2 - p_3 = 0$$
$$2(\cancel{1} + \cancel{t^2}) - (\cancel{2} + \cancel{3t} + \cancel{t^2}) - (\cancel{-3t} + \cancel{t^2}) = 0!$$

now, with coordvectors instead the "old way"

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$p_1(t) = 1 \cdot 1 + 0t + 1 \cdot t^2$

$p_2(t) = 2 + 3t + t^2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 3 & -3 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$c_1 = -2c_3$
$c_2 = c_3$
$c_3 = \text{free}$

$$c_3 = 1: \quad -2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

<u>Theorem</u>: If $T: V \to W$ is an isomorphism, then $\{\vec{v}_1, \ldots \vec{v}_p\}$ is (in)dependent in V if and only if $\{T\vec{v}_1, T\vec{v}_2 \ldots T\vec{v}_p\}$ are (in)dependent in W

<u>proof</u>: If $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p = \vec{0}$

then $T(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p) = T(\vec{0}) = \vec{0}$

linearity $\longrightarrow$ $\parallel$

$$c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \ldots c_p T(\vec{v}_p) = \vec{0}$$

So, if $\{\vec{v}_1, \ldots \vec{v}_p\}$ are dependent then $\{T(\vec{v}_1), T(\vec{v}_2) \ldots T(\vec{v}_p)\}$ are also dependent, with the same weights

apply same reasoning in reverse, using T^{-1}, to show that if $\{T\vec{v}_1, T\vec{v}_2, \ldots T\vec{v}_p\}$ are dependent in W, then $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p\}$ are dependent in V. ∎

by logic: same fact for "independent"

Exercise 3 Generalize the example of Exercise 1: Suppose $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$ is a non-standard basis of $\mathbb{R}^n$. And let $E = \{\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2, \ldots \underline{\boldsymbol{e}}_n\}$ be the standard basis of $\mathbb{R}^n$. For $\underline{\boldsymbol{x}} \in \mathbb{R}^n$, how do you convert between $\underline{\boldsymbol{x}}$ $= [\underline{\boldsymbol{x}}]_E$, and $[\underline{\boldsymbol{x}}]_\beta$, and vise-verse?

$*$ if $\vec{x} = [\vec{x}]_E = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

that means $[\vec{x}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$*$ $[\vec{x}]_E = B\,[\vec{x}]_\beta$ B is called $\underset{E \leftarrow \beta}{P}$

$**$ $B^{-1}[\vec{x}]_E = [\vec{x}]_\beta$ B^{-1} is called $\underset{\beta \leftarrow E}{P}$