Fri Mar 2
  • 4.6 The four subspaces associated with a matrix. the rank of a matrix.

Announcements:
  Finish Tuesday notes
  I think we'll finish Wed, but if we don't, that's fine.
  *fft

Warm-up Exercise:
  See Tuesday notes
Theorem Let $V$, $W$ be vector spaces, and $T : V \to W$ a linear transformation. If $T$ is 1-1 and onto, then the inverse function $T^{-1}$ is also a linear transformation, $T^{-1} : W \to V$. In this case, we call $T$ an isomorphism.

proof: We have to check that for all $\mathbf{u}, \mathbf{w} \in W$ and all $c \in \mathbb{R}$,

\[
T^{-1}(\mathbf{u} + \mathbf{w}) = T^{-1}(\mathbf{u}) + T^{-1}(\mathbf{w})
\]

\[
T^{-1}(c \mathbf{u}) = c T^{-1}(\mathbf{u}).
\]

Since $T$ is 1-1, it suffices to check that $T$ obeys the linearity property:

\[
T(\mathbf{u} + \mathbf{w}) = T(\mathbf{u}) + T(\mathbf{w})
\]

for scalar multiplication:

\[
T(c \mathbf{u}) = c T(\mathbf{u}).
\]

Theorem Let $V$ be a vector space, with basis $\beta = \{b_1, b_2, \ldots, b_n\}$. Then the coordinate transform $T : V \to \mathbb{R}^n$ defined by

\[T(\mathbf{v}) = [\mathbf{v}]_\beta\]

is linear, and it is an isomorphism.
Exercise 2: Use coordinates with respect to the basis \( \{1, t, t^2\} \), to check whether or not the set of polynomials \( \{ p_1(t), p_2(t), p_3(t) \} \) is a basis for \( P_2 \), where

\[
\begin{align*}
p_1(t) &= 1 + t^2 \\
p_2(t) &= 2 + 3t + t^2 \\
p_3(t) &= -3t + t^2.
\end{align*}
\]

i.e. check whether the coordinate vectors in \( \mathbb{R}^3 \) are independent & span \( \mathbb{R}^3 \). Then make conclusions about \( \{ p_1, p_2, p_3 \} \)

\[
\begin{align*}
p_2 &= 2p_1 - p_3 \\
p_2 - p_3 &= 0 \\
2p_1 - p_2 - p_3 &= 0.
\end{align*}
\]

We now, with coordinate vectors instead the "old way"

\[
\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}
\]

**Theorem:** If \( T : V \rightarrow W \) is an isomorphism, then

\( \{ \overrightarrow{v}_1, \ldots, \overrightarrow{v}_p \} \) is (in)dependent in \( V \)

if and only if

\( \{ T(\overrightarrow{v}_1), T(\overrightarrow{v}_2), \ldots, T(\overrightarrow{v}_p) \} \) are (in)dependent in \( W \)

**proof:** If \( c_1 \overrightarrow{v}_1 + c_2 \overrightarrow{v}_2 + \ldots + c_p \overrightarrow{v}_p = \overrightarrow{0} \)

then \( T(c_1 \overrightarrow{v}_1 + c_2 \overrightarrow{v}_2 + \ldots + c_p \overrightarrow{v}_p) = T(\overrightarrow{0}) = \overrightarrow{0} \)

by linearity

\[ c_1 T(\overrightarrow{v}_1) + c_2 T(\overrightarrow{v}_2) + \ldots + c_p T(\overrightarrow{v}_p) = \overrightarrow{0} \]

So, if \( \{ \overrightarrow{v}_1, \ldots, \overrightarrow{v}_p \} \) are dependent then \( \{ T(\overrightarrow{v}_1), T(\overrightarrow{v}_2), \ldots, T(\overrightarrow{v}_p) \} \) are also dependent, with the same weights

for "independent" apply same reasoning in reverse, using \( T^{-1} \), to show that if \( \{ T \overrightarrow{v}_1, T \overrightarrow{v}_2, \ldots, T \overrightarrow{v}_p \} \) are dependent in \( W \), then \( \{ \overrightarrow{v}_1, \overrightarrow{v}_2, \ldots, \overrightarrow{v}_p \} \) are dependent in \( V \).
Exercise 3  Generalize the example of Exercise 1: Suppose $\beta = \{b_1, b_2, \ldots, b_n\}$ is a non-standard basis of $\mathbb{R}^n$. And let $E = \{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, how do you convert between $x = [x]_E$, and $[x]_\beta$, and vise-verse?

$\star$ \quad \text{if } \bar{x} = [x]_E = c_1 b_1 + c_2 b_2 + \ldots + c_n b_n = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

that means $[x]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$\star$ \quad $[x]_E = B [x]_\beta$ \quad \text{B is called } \begin{bmatrix} P \end{bmatrix}_{E \leftarrow \beta}$

$\star \star$ \quad $B^{-1} [x]_E = [x]_\beta$ \quad \text{B}^{-1} \text{ is called } \begin{bmatrix} P \end{bmatrix}_{\beta \leftarrow E}$