Similar matrices. This generalizes the way in which diagonalizable matrices are similar to diagonal ones:

Definition The $n \times n$ matrices A, B are said to be "*similar*" if there is and invertible matrix P so that
$$P^{-1} A P = B.$$

$P^{-1}AP = D$ diagonalizable

Notice that *being similar* is an *equivalence relation*:

1) If A is similar to B with the matrix P, then B is similar to A, with the matrix P^{-1} :

P $\quad P^{-1}$

$$P^{-1} A P = B \quad \Rightarrow \quad A = P B P^{-1}.$$

2) A is similar to itself, with $P = I$:

$$A = I^{-1} A I$$

3) Being similar is *transitive*: if A is similar to B and B is similar to C, then A is similar to C: If we have invertible matrices P, Q so that

$$P^{-1} A P = B$$
$$Q^{-1} B Q = C$$

then

$$Q^{-1} P^{-1} A P Q = Q^{-1} B Q = C.$$

so A is similar to C via the matrix PQ.

wikipedia.

These three "equivalence relations" mean that the space all $n \times n$ matrices can be *partitioned* into subsets of matrices which are similar to each other.

We'll see tomorrow that *similar matrices* represent the *same* linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, but with the matrices expressed with respect to different bases. For now (and for one of your homework problems tomorrow), we need to know that

Theorem Let A and B be similar matrices. Then they have the same characteristic polynomial, so the same eigenvalues. (They won't have the same eigenvectors, though.) same characteristic polynomial

proof Let

$$P^{-1} A P = B.$$

Then

$$det(B - \lambda I) = det(P^{-1} A P - \lambda I)$$

$$= det \left(P^{-1} A P - \lambda P^{-1} I P\right) \qquad P^{-1}(AP - \lambda I P)$$

$$= det \left(P^{-1} (A - \lambda I) P\right) \qquad P^{-1}(A - \lambda I) P$$

$$= \underline{\det\left(P^{-1}\right)}\ \underline{\det\left(A - \lambda I\right)}\ \underline{\det\left(P\right)}$$

$$= \underline{\underline{\det\left(A - \lambda I\right).}}$$

$P^{-1} P = I$

$\det(P^{-1}) \det(P) = \det I$

QED

$= 1$

Wed Mar 14 not on midterm, but ties together a lot of ideas on test

- 5.4 Similar matrices and the matrix of a linear transformation with respect to bases

you can keep hw until Fri. if you want.

Announcements: no quiz today

midterm: 3.1-3.3 determinants
4.1-4.7 vector spaces
[5.1-5.3 eigenvalues, eigenvectors, diagonalizability

- practice midterm later today
- review session tomorrow 12:55-2:20 JWB 335

Warm-up Exercise: Find eigenvalues and eigenspace bases

a) for $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

b) Find P so that $P^{-1}AP = D$, where D is diagonal

a) $|A-\lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (\lambda-3)^2 - 1 = (\lambda-3-1)(\lambda-3+1)$
$= (\lambda-4)(\lambda-2) = 0$

$\lambda = 2, 4.$

$E_{\lambda=2}$

$(A-2I)\vec{v}$

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$

$E_{\lambda=2} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$

$E_{\lambda=4}$

$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$

$E_{\lambda=4} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}.$

b). $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$AP = PD$
$P^{-1}AP = D$

$AP = A[\vec{v}_1 | \vec{v}_2]$
$= [\lambda_1\vec{v}_1 | \lambda_2\vec{v}_2]$
$= [\vec{v}_1 | \vec{v}_2]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$
$= PD$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} \underbrace{A P}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$P^{-1} \begin{bmatrix} 2 & 4 \\ -2 & 4 \end{bmatrix}$$

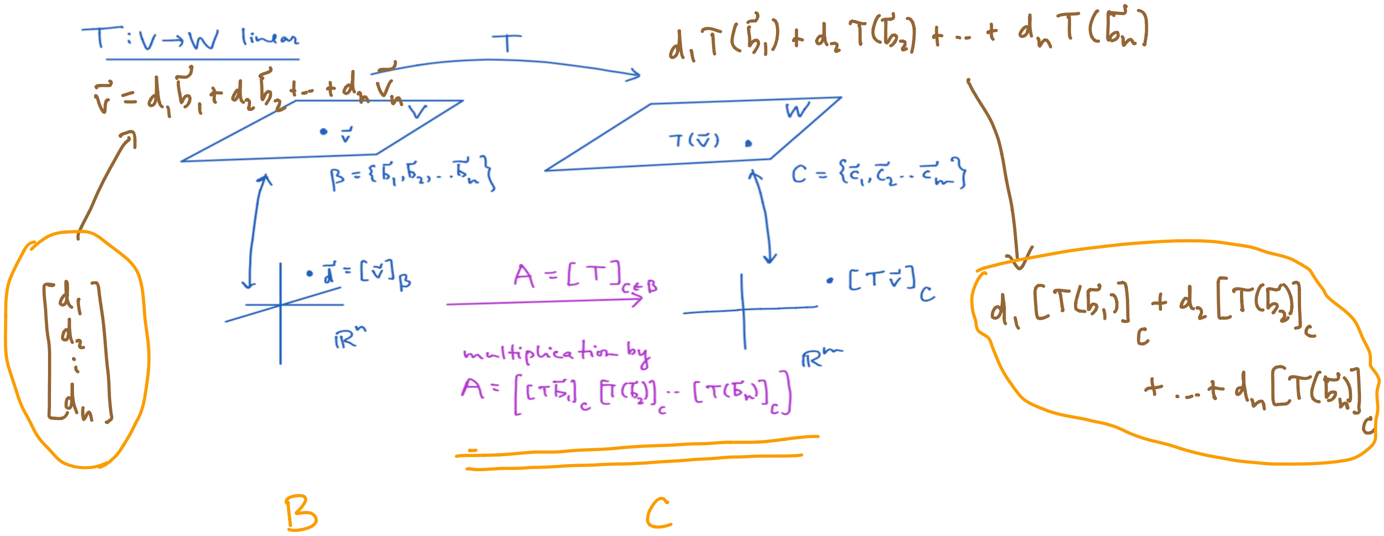
$$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

$$|A - \lambda I| = |-(\lambda I - A)|$$

$$|A - \lambda I| = (-1)^n |\lambda I - A| = 0$$

If we have a linear transformation $T : V \to W$ and bases $B = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \dots \underline{\boldsymbol{b}}_n\}$ in V, $C = \{\underline{\boldsymbol{c}}_1, \underline{\boldsymbol{c}}_2, \dots \underline{\boldsymbol{c}}_m\}$ in W, then the matrix of T with respect to these two bases transforms the B coordinates of vectors $\underline{\boldsymbol{v}} \in V$ to the C coordinates of $T(\underline{\boldsymbol{v}})$ in a straightforward way:



Exercise 1) Let $V = P_3 = span\{1, t, t^2, t^3\}$, $W = P_2 = span\{1, t, t^2\}$, and let $D : V \to W$ be the derivative operator. Find the matrix of D with respect to the bases $\{1, t, t^2, t^3\}$ in V and $\{1, t, t^2\}$ in W. Test your result.

$$A = \begin{bmatrix} [D(1)]_C & [D(t)]_C & [D(t^2)]_C & [D(t^3)]_C \end{bmatrix}$$

$D(1) = 0$

$D(t) = 1$

$D(t^2) = 2t$

$D(t^3) = 3t^2$

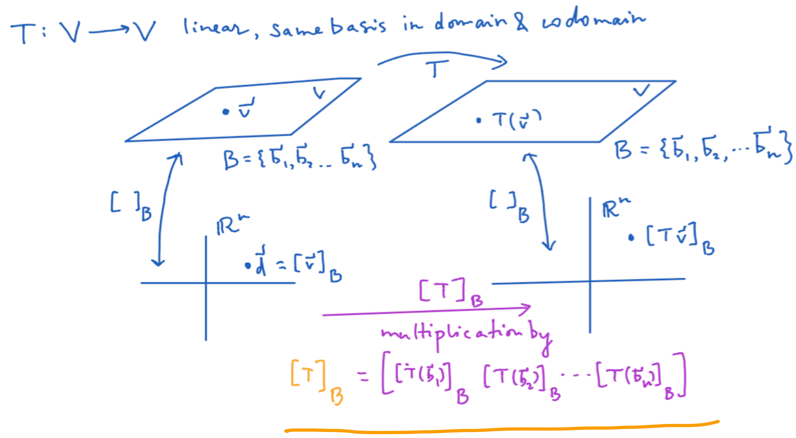
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 3 \end{bmatrix} \leftarrow [p'(t)]_C$$

checks!

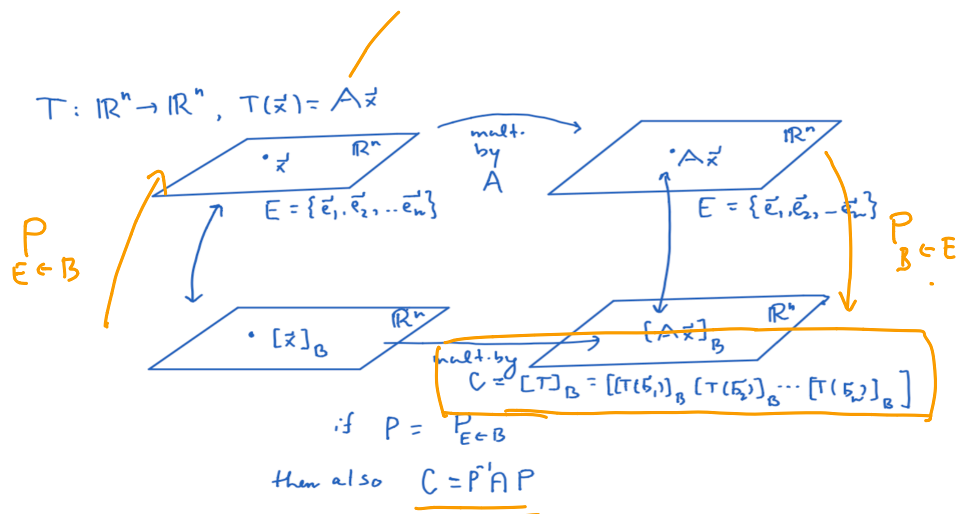
$$D(\underbrace{t^3 + 5t^2 + 2t + 10}_{p(t)}) = 3t^2 + 10t + 2$$

$$[p(t)]_B = \begin{bmatrix} 10 \\ 2 \\ 5 \\ 1 \end{bmatrix}$$

A special case of the matrix for a linear transformation is when $T : V \to V$ and one uses the same basis in the domain and codomain:



And a special case of that is when $T : \mathbb{R}^n \to \mathbb{R}^n$ is a matrix transformation $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$, and we find the matrix of T with respect to a non-standard basis. This is how similar matrices arise: as descriptions of the same linear transformation, but using different bases:



A special case of similar matrices is when A is diagonalizable and P is a matrix whose columns are an eigenbasis for $\mathbb{R}^n$. Then C is a diagonal matrix with the corresponding eigenvalues in each column,

Example: $T(\vec{x}) = A\vec{x} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\vec{x}$ from warmup problem

A is diagonalizable, $E_{\lambda=4} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$, $E_{\lambda=2} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

Let $B = \left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ be non-standard basis of $\mathbb{R}^2$

$[T]_B = \left[\,[T(\vec{b}_1)]_B \;\; [T(\vec{b}_2)]_B\,\right] = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = D$ since $T(\vec{b}_1) = 2\vec{b}_1$, $T(\vec{b}_2) = 4\vec{b}_2$

in other words, using the eigenvector basis the matrix for T is the diagonal matrix

$$[T]_B = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = D$$

and the algebraic fact that for $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$,

$$P^{-1} A P = D$$

corresponds to the orange arrows on the diagram:

$$[T]_B = \underset{B \leftarrow E}{P} \; A \; \underset{E \leftarrow B}{P}$$

converts B-coords of input vectors into B-coords of output vectors

B coords to E coords

matrix for T using E coords

convert E coords of output back to B coords