There are situations where we are guaranteed a basis of $\mathbb{R}^n$ made out eigenvectors of A:

Theorem 1: Let A be an $n \times n$ matrix with distinct real eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$. Let $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ be corresponding (non-zero) eigenvectors, $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$. Then the set

$$\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$$

is linearly independent, and so is a basis for $\mathbb{R}^n$.....this is one we can prove!

nifty proof. Assume the vectors in the set are dependent. (we'll end up with a contradiction)

$\{\vec{v}_1\}$ independent

$\{\vec{v}_1, \vec{v}_2\}$?

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

⋮

$\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ assume dependent

Let $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p\}$ be the first dependent set in this list. Since its the first (shortest) set $\vec{v}_p$ is a combo of the earlier $\vec{v}_j$'s

(1) $$\boxed{\vec{v}_p = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_{p-1}\vec{v}_{p-1}}$$

not all c_j's $= 0$

$$\Rightarrow A\vec{v}_p = A(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_{p-1}\vec{v}_{p-1})$$

$$\lambda_p\vec{v}_p = c_1 A\vec{v}_1 + c_2 A\vec{v}_2 + \ldots + c_{p-1}A\vec{v}_{p-1}$$

(2) $$\boxed{\lambda_p\vec{v}_p = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \ldots + c_{p-1}\lambda_{p-1}\vec{v}_{p-1}}$$

from (2), $\lambda_p \neq 0$, — if it was, $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_{p-1}\}$ is depend.

(3) $$\boxed{\Rightarrow \vec{v}_p = c_1\frac{\lambda_1}{\lambda_p}\vec{v}_1 + c_2\frac{\lambda_2}{\lambda_p}\vec{v}_2 + \ldots + c_{p-1}\frac{\lambda_{p-1}}{\lambda_p}\vec{v}_{p-1}}$$

Eqtn 1 − Eqtn 3 :

$$\vec{0} = c_1\left(1-\frac{\lambda_1}{\lambda_p}\right)\vec{v}_1 + c_2\left(1-\frac{\lambda_2}{\lambda_p}\right)\vec{v}_2 + \ldots + c_{p-1}\left(1-\frac{\lambda_{p-1}}{\lambda_p}\right)\vec{v}_p$$

no $\frac{\lambda_j}{\lambda_p} = 1$ on RHS,

showed $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_{p-1}\}$ is also dependent !!

⋇.

Theorem 2

Let $A_{n \times n}$ have factored characteristic polynomial

algebraic multiplicity

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \ldots (\lambda - \lambda_m)^{k_m}$$

where like terms have been collected so that each λ_j is distinct (i.e different). Notice that

$$k_1 + k_2 + \ldots + k_m = n$$

because the degree of $p(\lambda)$ is n.

geometric multiplicity.

- Then $1 \le dim\left(E_{\lambda = \lambda_j}\right) \le k_j$. If $dim\left(E_{\lambda = \lambda_j}\right) < k_j$ then the λ_j eigenspace is called defective.
- The matrix A is diagonalizable if and only if each $dim\left(E_{\lambda = \lambda_j}\right) = k_j$. In this case, one obtains an $\mathbb{R}^n$

eigenbasis simply by combining bases for each eigenspace into one collection of n vectors. (The same definitions and reasoning can apply to complex eigenvalues and eigenvectors, and a basis of $\mathbb{C}^n$.)

(The proof of this theorem is fairly involved. It was illustrated in a positive way by Exercise 2, and in a negative way by Exercise 3.)

yesterday B that was diagonalizable.

$$|B - \lambda I| = -(\lambda-2)^2(\lambda-3)$$

$$\dim E_{\lambda=2} = 2$$

$$\dim E_{\lambda=1} = 1$$

warmup today C was not diagonalizable

$$|C - \lambda I| = -(\lambda-2)^2(\lambda-3)$$

$\dim E_{\lambda=2} = 1$ $\leftarrow$ defective

Tues Mar 13

- 5.3 *Diagonalizable* matrices and *Similar matrices*.

Announcements: ⊙ office hours today canceled.
⊙ I'll try this afternoon to put up a practice test

'til 12:57

Warm-up Exercise: Find all eigenvalues (2, 3), and eigenspace bases for

$$C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

for triangular matrix, diag. elts are eigenvalues:

$$C - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} \qquad \det = (2-\lambda)^2(3-\lambda)$$

$E_{\lambda=2} = \operatorname{Nul}(C-2I)$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$E_{\lambda=2} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

algebraic multiplicity of $\lambda = 2$, is 2, the power of $(\lambda-2)$ in $p(\lambda)$

$p(\lambda) = -(\lambda-2)^2(\lambda-3)$

but $E_{\lambda=2}$ is only 1-dim'l ☹

$E_{\lambda=3} = \operatorname{Nul}\, C-3I$

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$E_{\lambda=3} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

long way: $v_1 = 0$, $v_2 = 0$, $v_3 = t$ free $\quad \vec{v} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

we only have 2 independent eigenvectors

so, no basis of $\mathbb{R}^3$ made out eigenvectors for C.

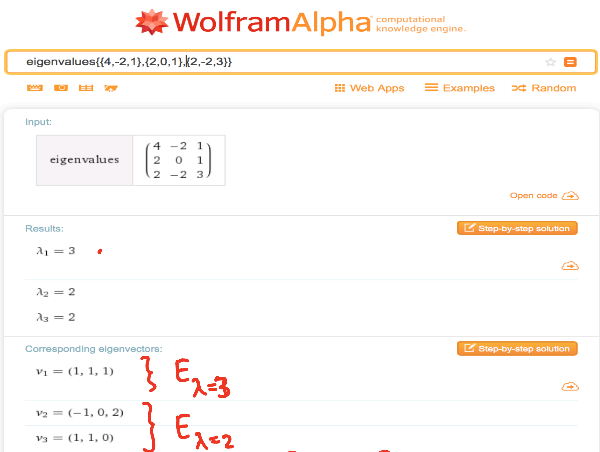
Continuing with the example from yesterday ...

If, for the matrix $A_{n \times n}$, there is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A, then we can understand the geometry of the transformation

$$T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$$

almost as well as if A is a diagonal matrix, and so we call such matrices *diagonalizable*. Having such a basis of eigenvectors for a given matrix is also extremely useful for algebraic computations, and will give another reason for the word *diagonalizable* to describe such matrices.

Use an $\mathbb{R}^3$ basis made of out eigenvectors of the matrix B in Exercise 2, yesterday, and put them into the columns of a matrix we will call P. We could order the eigenvectors however we want, but we'll put the $E_{\lambda=2}$ basis vectors in the first two columns, and the $E_{\lambda=3}$ basis vector in the third column:



$\} E_{\lambda=3}$

$\} E_{\lambda=2}$

$$B = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\vec{v}_2 + \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

in $E_{\lambda=2}$ $\quad E_{\lambda=3}$

$$P := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

Now do algebra (check these steps and discuss what's going on!)

$$\begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix} \left[\begin{array}{c|c|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{array}\right]$$

B

$$= \left[\begin{array}{c|c|c} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 4 & -4 & 3 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{array}\right] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

In other words,

$$B\,P = P\,\mathrm{D},$$

$B = PDP^{-1}$ • mult by P^{-1} on the right

$P^{-1}BP = D$ • mult by P^{-1} on the left

where D is the diagonal matrix of eigenvalues (for the corresponding columns of eigenvectors in P).
Equivalently (multiply on the right by P^{-1} or on the left by P^{-1}):

$$B = P\,\mathrm{D}\,P^{-1} \text{ and } P^{-1}BP = \mathrm{D}.$$

Exercise 1) Use one of the the identities above to show how B^{100} can be computed with only two matrix multiplications!

$$B^{100} = \underbrace{PD\overbrace{P^{-1}\,\underbrace{P}}^{I}D\overbrace{P^{-1}\,\underbrace{P}}^{I}DP^{-1}}\cdots\underbrace{PDP^{-1}} \quad \text{100 times}$$

$$= PD^{100}P^{-1}$$

$$\begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}^{100}$$

$$\begin{bmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & a_3 \end{bmatrix}\begin{bmatrix} b_1 & & 0 \\ & b_2 & \\ 0 & & b_3 \end{bmatrix} = \begin{bmatrix} a_1b_1 & 0 & 0 \\ 0 & a_2b_2 & 0 \\ 0 & 0 & a_3b_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2^{100} & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 3^{100} \end{bmatrix}\begin{bmatrix} & P^{-1} & \end{bmatrix}$$

Definition: Let $A_{n \times n}$. If there is an $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ consisting of eigenvectors of A, then A is called diagonalizable. This is precisely why:

Write $A\, \underline{\boldsymbol{v}}_j = \lambda_j\, \underline{\boldsymbol{v}}_j$ (some of these λ_j may be the same, as in the previous example). Let P be the matrix

$$P = \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right].$$

Then, using the various ways of understanding matrix multiplication, we see

$$A\,P = A\left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right] = \left[\lambda_1 \underline{\boldsymbol{v}}_1 \middle| \lambda_2 \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \lambda_n \underline{\boldsymbol{v}}_n\right]$$

$$= \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ : & : & \ldots & : \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

$$A\,P = P\,\mathrm{D}$$
$$A = P\,\mathrm{D}\,P^{-1}$$
$$P^{-1} A\,P = \mathrm{D}\,.$$

Unfortunately, as we've already seen, not all matrices are diagonalizable:

Exercise 2) Show that

$$C := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is not diagonalizable. (Even though it has the same characteristic polynomial as B, which was diagonalizable.

This was your Tuesday warm-up exercise

Similar matrices. This generalizes the way in which diagonalizable matrices are similar to diagonal ones:

<u>Definition</u> The $n \times n$ matrices A, B are said to be *similar* if there is and invertible matrix P so that

$$P^{-1} A P = B.$$

Notice that *being similar* is an *equivalence relation*:

1) If A is similar to B with the matrix P, then B is similar to A, with the matrix P^{-1} :

$$P^{-1} A P = B \quad \Rightarrow \quad A = P B P^{-1}.$$

2) A is similar to itself, with $P = I$:

$$A = I^{-1} A I$$

3) Being similar is *transitive*: if A is similar to B and B is similar to C, then A is similar to C: If we have invertible matrices P, Q so that

$$P^{-1} A P = B$$
$$Q^{-1} B Q = C$$

then

$$Q^{-1} P^{-1} A P Q = Q^{-1} B Q = C.$$

so A is similar to C via the matrix PQ.

These three "equivalence relations" mean that the space all $n \times n$ matrices can be *partitioned* into subsets of matrices which are similar to each other.

We'll see tomorrow that *similar matrices* represent the *same* linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, but with the matrices expressed with respect to different bases. For now (and for one of your homework problems tomorrow), we need to know that

<u>Theorem</u> Let A and B be similar matrices. Then they have the same characteristic polynomial, so the same eigenvalues. (They won't have the same eigenvectors, though.)

<u>proof</u> Let

$$P^{-1} A P = B.$$

Then

$$det(B - \lambda I) = det(P^{-1} A P - \lambda I)$$

$$= det \left(P^{-1} A P - \lambda P^{-1} I P\right)$$

$$= det \left(P^{-1} \left(A - \lambda I\right) P\right)$$

$$= det\left(P^{-1}\right)\, det\left(A - \lambda\, I\right)\, det\,(P)$$

$$= det\left(A - \lambda\, I\right).$$

QED