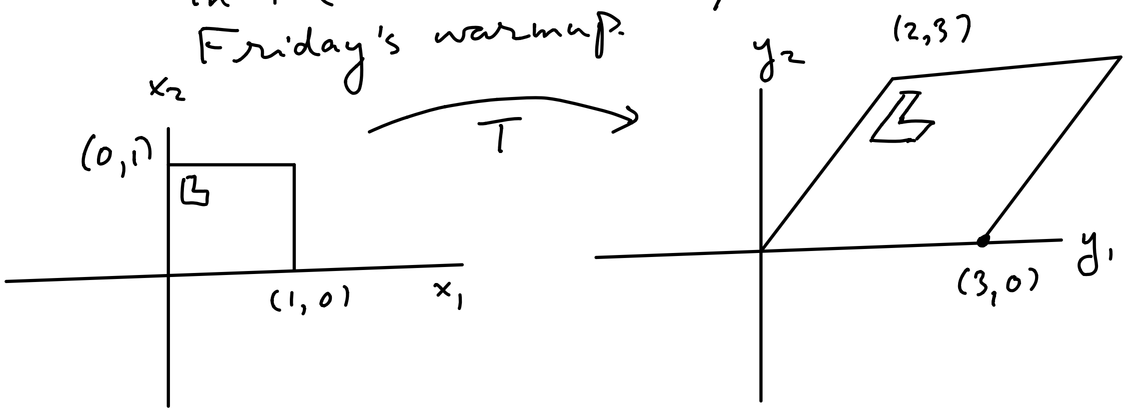
Math 2270-004 Week 10 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 5.2-5.4

Mon Mar 12
- 5.2 matrix eigenspaces

Announcements:
- Office hours today are cancelled.
- I might post some review material later today for midterm 2 Friday
- Review session Th 12:55 – 2:20 JWB 335.
- Long HW 2nd assignment the week after S.B. (using characteristic polynomial)

'til 10:57

Warm-up Exercise: Find all eigenvalues and eigenspace bases for $A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$. This was the matrix in the transformation you thought about in Friday's warmup.



$A\vec{v} = \lambda \vec{v} \quad (\vec{v} \neq \vec{0})$

$A\vec{v} - \lambda\vec{v} = \vec{0}$

$(A - \lambda I)\vec{v} = \vec{0}$

non-zero solns for given λ iff $\det(A - \lambda I) = 0$

$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0$

$\lambda = 3.$

$E_{\lambda=3}$

$(A - 3I)\vec{v} = \vec{0}$

$\left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$1 \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$E_{\lambda=3} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ $\quad \left\{\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is eigenbasis.

or backsolve $\quad v_1 = t, \; v_2 = 0 \quad \vec{v} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

Monday Review!

We've been studying *linear transformations* $T : V \rightarrow W$ between vector spaces, which are generalizations of matrix transformations $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ given as $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$.

We've been studying how coordinates change when we change bases for finite dimensional vector spaces V.

On Friday we introduced the notion of *eigenvectors* for linear transformations $T : V \rightarrow V$, non-zero vectors $\underline{\boldsymbol{v}}$ so that T transforms $\underline{\boldsymbol{v}}$ to a multiple of itself. This multiple λ is called the *eigenvalue* of $\underline{\boldsymbol{v}}$. In other words, $T(\underline{\boldsymbol{v}}) = \lambda\,\underline{\boldsymbol{v}}$.

For our eigenvector discussion we're focusing on $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$. In this case we talk about eigenvectors of the matrix A, with eigenvalue λ, $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$. The idea is that because eigenvectors are transformed in just about the most simple way possible by the matrix, they will give us computational and conceptual ~~insite~~ insight into the matrix transformation. We'll see how this plays out.

On Friday we introduced the characteristic equation method of finding eigenvalues of a matrix first, and then the eigenvectors (eigenspace bases) second:

How to find eigenvalues and eigenvectors (eigenspace bases) systematically:

If

$$A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$$

$$\Leftrightarrow A\,\underline{\boldsymbol{v}} - \lambda\,\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$$

$$\Leftrightarrow A\,\underline{\boldsymbol{v}} - \lambda\, I\,\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$$

where I is the identity matrix.

$$\Leftrightarrow (A - \lambda\, I)\underline{\boldsymbol{v}} = \underline{\mathbf{0}}\,.$$

As we know, this last equation can have non-zero solutions $\underline{\boldsymbol{v}}$ if and only if the matrix $(A - \lambda\, I)$ is not invertible, i.e.

$$\Leftrightarrow \underline{det(A - \lambda\, I) = 0}\,.$$

So, to find the eigenvalues and eigenvectors of matrix you can proceed as follows:

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} \quad \text{degree 2.}$$

<u>Step 1</u>

- Compute the polynomial in λ

$$p(\lambda) = det(A - \lambda\, I)\,.$$

If $A_{n\times n}$ then $p(\lambda)$ will be degree n. This polynomial is called the <u>characteristic polynomial</u>. The eigenvalues will be the <u>roots λ_j</u> of the <u>characteristic equation</u>

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$$

degree 3 in λ

etc.

$$det(A - \lambda\, I) = 0.$$

- For each such root λ_j, the homogeneous solution space of vectors $\underline{\boldsymbol{v}}$ solving

$$(A - \lambda_j\, I)\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$$

i.e

$$Nul\ (A - \lambda_j\, I)$$

will be the sub vector space of eigenvectors with eigenvalue λ_j. This subspace of eigenvectors will be at least one dimensional, since $(A - \lambda_j\, I)$ does not reduce to the identity and so the explicit homogeneous solutions will have free parameters. Find a basis of eigenvectors for this subspace. Follow this procedure for each eigenvalue, i.e. for each root of the characteristic polynomial.

<u>Notation</u>: The subspace of eigenvectors for eigenvalue λ_j is called the <u>λ_j eigenspace</u>, and we'll denote it by $E_{\lambda=\lambda_j}$. The basis of eigenvectors is called an <u>eigenbasis</u> for $E_{\lambda=\lambda_j}$.

We did part a on Friday, but didn't get to part b:

Exercise 1) a) Use the systematic algorithm to find the eigenvalues and eigenbases for the non-diagonal matrix

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

b) Use your work to describe the geometry of the linear transformation in terms of directions that get stretched:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

1a)

$$A - \lambda I = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{bmatrix}$$

$$p(\lambda) = \begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda-3)\cdot(\lambda-2) - 2 = \lambda^2 - 5\lambda + 4 = (\lambda-1)\cdot(\lambda-4).$$

So the eigenvalues of A are $\lambda = 1, 4$

$E_{\lambda=4}$: $Nul\ (A - 4I)$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & -2 & 0 \end{array}\right] \qquad E_{\lambda=4} = span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$$

$E_{\lambda=1}$: $Nul\ (A - I)$

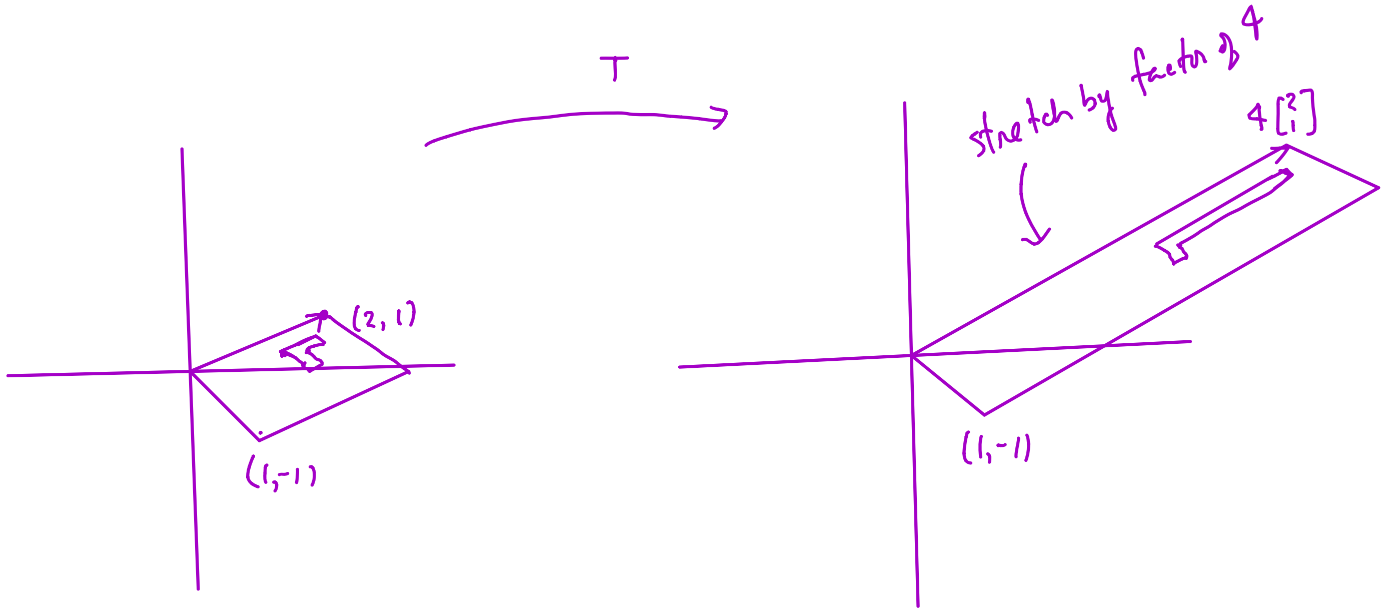
$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 1 & 1 & 0 \end{array}\right] \qquad E_{\lambda=1} = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

Let's do part b!

b) Use the eigenspace information to describe the geometry of the linear transformation in terms of directions that get stretched.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \qquad E_{\lambda=4} = span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\} \qquad E_{\lambda=1} = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}.$$

$$\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



<u>Exercise 2)</u> Find the eigenvalues and eigenspace bases for

$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}.$$

(i) Find the characteristic polynomial and factor it to find the eigenvalues. $(p(\lambda) = -(\lambda-2)^2(\lambda - \cancel{1}\,3))$ typo

(ii) for each eigenvalue, find bases for the corresponding eigenspaces.

(iii) Can you describe the transformation $T(\underline{\mathbf{x}}) = B\underline{\mathbf{x}}$ geometrically using the eigenbases? Does $det(B)$ have anything to do with the geometry of this transformation?

$$\begin{vmatrix} 4-\lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 2 & -2 & 3-\lambda \end{vmatrix}$$

<u>expand → cubic → factor.</u>

"cheat"

$$\begin{vmatrix} 4-\lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 0 & \lambda-2 & 2-\lambda \end{vmatrix} \qquad -R_2+R_3 \to R_3$$

$$= (\lambda-2)\begin{vmatrix} 4-\lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= (\lambda-2)\begin{vmatrix} 4-\lambda & -1 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 1 & -1 \end{vmatrix} \qquad \begin{matrix} R_1+R_3 \to R_1 \\ R_2+R_3 \to R_2 \end{matrix}$$

$$= (\lambda-2)\left[0 - 0 - 1\begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix}\right.$$

$$= (\lambda-2)(-1)\,\left((\lambda-4)(\lambda-1)+2\right)$$

$$= -(\lambda-2)\left[\lambda^2 - 5\lambda + 6\right]$$

$$= -(\lambda-2)\left(\lambda-2)(\lambda-3)\right)$$

$$= -(\lambda-2)^2(\lambda-3).$$

e.vals $\lambda = 2, 3.$

$$(4-\lambda)(1-\lambda) = -(\lambda-4)(-)(\lambda-1) = (\lambda-4)(\lambda-1)$$

$E_{\lambda=2}$:

$(B-2I)\vec{v} = \vec{0}$

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$E_{\lambda=2}$: $\operatorname{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-2\end{bmatrix}\right\}.$

rref:

$$\left[\begin{array}{ccc|c} 1 & -1 & .5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 = t_2 - .5t_3$

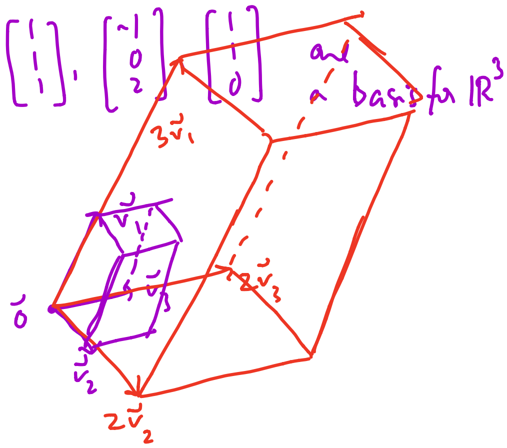
$v_2 = t_2$

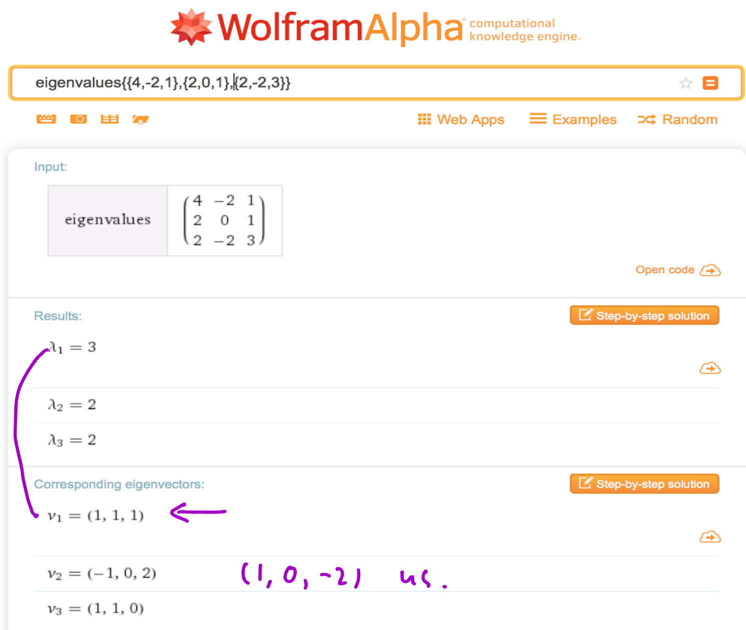
$v_3 = t_3$

$$\vec{v} = t_2\begin{bmatrix}1\\1\\0\end{bmatrix} + t_3\begin{bmatrix}-.5\\0\\1\end{bmatrix}$$

$E_{\lambda=3}$: skip (straight forward)

Your solution will be related to the output below:





In most of our examples so far, it turns out that by collecting bases from each eigenspace for the matrix $A_{n \times n}$, and putting them together, we get a basis for $\mathbb{R}^n$. This lets us understand the <u>geometry</u> of the transformation

$$T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$$

almost as well as if A is a diagonal matrix. It does not always happen that the matrix A an basis of $\mathbb{R}^n$ made consisting of eigenvectors for A. (Even when all the eigenvalues are real.) When it does happen, we say that A is <u>diagonalizable.</u> Here's an example of a matrix which is NOT diagonalizable:

<u>Exercise 3</u>: Find matrix eigenvalues and eigenspace basis for each eigenvalue, for

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}.$$

Explain why there is no basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

warmup