We will not necessarily finish the material from a given day’s notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 5.2-5.4

Mon Mar 12
- 5.2 matrix eigenspaces

Announcements:
- Office hours today are cancelled.
- I might post some review material later today for midterm 2.
- Review session Thu 12:55-2:20
- Long HW 2nd JWB 335 assignment the week after S:B.

Warm-up Exercise:
'4.1 10:57

Find all eigenvalues and eigenspace bases for $A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$. This was the matrix in the transformation you thought about in Friday's warmup.

\[
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}
\]

\[
A \vec{v} = \lambda \vec{v} \quad (\vec{v} \neq \vec{0})
\]

\[
(A - \lambda I) \vec{v} = \vec{0}
\]

non-zero solutions for given $\lambda$ iff $\det(A - \lambda I) = 0$

\[
\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0
\]

$\lambda = 3$.

$E_{\lambda = 3} = \text{span}\{[1], [0]\}$

\[
E_{\lambda = 3} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

basis.

\[
\text{or backsolve } \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]
Monday Review!

We've been studying linear transformations $T : V \rightarrow W$ between vector spaces, which are generalizations of matrix transformations $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ given as $T(x) = Ax$.

We've been studying how coordinates change when we change bases for finite dimensional vector spaces $V$.

On Friday we introduced the notion of eigenvectors for linear transformations $T : V \rightarrow V$, non-zero vectors $v$ so that $T$ transforms $v$ to a multiple of itself. This multiple $\lambda$ is called the eigenvalue of $v$. In other words, $T(v) = \lambda v$.

For our eigenvector discussion we're focusing on $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $T(x) = Ax$. In this case we talk about eigenvectors of the matrix $A$, with eigenvalue $\lambda$, $A v = \lambda v$. The idea is that because eigenvectors are transformed in just about the most simple way possible by the matrix, they will give us computational and conceptual insight into the matrix transformation. We'll see how this plays out.

On Friday we introduced the characteristic equation method of finding eigenvalues of a matrix first, and then the eigenvectors (eigenspace bases) second:
How to find eigenvalues and eigenvectors (eigenspace bases) systematically:

If

\[ A \mathbf{v} = \lambda \mathbf{v} \]

\[ \Leftrightarrow A \mathbf{v} - \lambda \mathbf{v} = \mathbf{0} \]

\[ \Leftrightarrow A \mathbf{v} - \lambda I \mathbf{v} = \mathbf{0} \]

where \( I \) is the identity matrix.

\[ \Leftrightarrow (A - \lambda I)\mathbf{v} = \mathbf{0} . \]

As we know, this last equation can have non-zero solutions \( \mathbf{v} \) if and only if the matrix \( (A - \lambda I) \) is not invertible, i.e.

\[ \Leftrightarrow \det(A - \lambda I) = 0 . \]

So, to find the eigenvalues and eigenvectors of matrix you can proceed as follows:

- **Step 1**
  - Compute the polynomial in \( \lambda \)

\[ p(\lambda) = \det(A - \lambda I) . \]

If \( A_{n \times n} \) then \( p(\lambda) \) will be degree \( n \). This polynomial is called the **characteristic polynomial**. The eigenvalues will be the roots \( \lambda \) of the characteristic equation

\[ \det(A - \lambda I) = 0 . \]

- For each such root \( \lambda \), the homogeneous solution space of vectors \( \mathbf{v} \) solving

\[ (A - \lambda \mathbf{I})\mathbf{v} = \mathbf{0} \]

i.e.

\[ \text{Nul} \ (A - \lambda \mathbf{I}) \]

will be the sub vector space of eigenvectors with eigenvalue \( \lambda \). This subspace of eigenvectors will be at least one dimensional, since \( (A - \lambda \mathbf{I}) \) does not reduce to the identity and so the explicit homogeneous solutions will have free parameters. Find a basis of eigenvectors for this subspace. Follow this procedure for each eigenvalue, i.e. for each root of the characteristic polynomial.

**Notation:** The subspace of eigenvectors for eigenvalue \( \lambda \) is called the \( \lambda \)-eigenspace, and we'll denote it by \( E_{\lambda = \lambda} \). The basis of eigenvectors is called an **eigenbasis** for \( E_{\lambda = \lambda} \).
We did part a on Friday, but didn't get to part b:

Exercise 1) a) Use the systematic algorithm to find the eigenvalues and eigenbases for the non-diagonal matrix

\[ A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}. \]

b) Use your work to describe the geometry of the linear transformation in terms of directions that get stretched:

\[ T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \]

1a)

\[ A - \lambda I = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3 - \lambda & 2 \\ 1 & 2 - \lambda \end{bmatrix}. \]

\[ p(\lambda) = \begin{vmatrix} 3 - \lambda & 2 \\ 1 & 2 - \lambda \end{vmatrix} = (\lambda - 3) \cdot (\lambda - 2) - 2 = \lambda^2 - 5 \lambda + 4 = (\lambda - 1) \cdot (\lambda - 4). \]

So the eigenvalues of \( A \) are \( \lambda = 1, 4 \)

\[ E_{\lambda=4} : \text{Nul} (A - 4 I) \]

\[ \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \]

\[ E_{\lambda=4} = \text{span} \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \]

\[ E_{\lambda=1} : \text{Nul} (A - 1 I) \]

\[ \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \]

\[ E_{\lambda=1} = \text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \]

Let's do part b!
b) Use the eigenspace information to describe the geometry of the linear transformation in terms of directions that get stretched.

\[ A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \]

\[ E_{\lambda=4} = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \quad E_{\lambda=1} = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} . \]

\[
\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \\
\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

\[(2, 1) \xrightarrow{T} (8, 4) \quad (1, -1) \xrightarrow{T} (1, -1) \]

Stretch by factor 4
Exercise 2) Find the eigenvalues and eigenspace bases for

\[ B := \begin{bmatrix}
4 & -2 & 1 \\
2 & 0 & 1 \\
2 & -2 & 3
\end{bmatrix}. \]

(i) Find the characteristic polynomial and factor it to find the eigenvalues. \( p(\lambda) = - (\lambda - 2)^2 (\lambda - 3) \)

(ii) For each eigenvalue, find bases for the corresponding eigenspaces.

(iii) Can you describe the transformation \( T(x) = Bx \) geometrically using the eigenbases? Does \( \det(B) \) have anything to do with the geometry of this transformation?

\[
\begin{vmatrix}
4 - \lambda & -2 & 1 \\
2 & -\lambda & 1 \\
2 & -2 & 3 - \lambda
\end{vmatrix}
= (\lambda - 2)
\begin{vmatrix}
4 - \lambda & -2 \\
2 & -\lambda
\end{vmatrix}
= (\lambda - 2)(4 - \lambda - 2 - \lambda + 2)
= \lambda^2 - 5\lambda + 6
= (\lambda - 2)(\lambda - 3)
= -(\lambda - 2)(\lambda - 3) \text{ (straight forward)}
\]

\[ \text{e.vals } \lambda = 2, 3. \]
In most of our examples so far, it turns out that by collecting bases from each eigenspace for the matrix $A_{n \times n}$, and putting them together, we get a basis for $\mathbb{R}^n$. This lets us understand the geometry of the transformation

$$T(x) = Ax$$

almost as well as if $A$ is a diagonal matrix. It does not always happen that the matrix $A$ an basis of $\mathbb{R}^n$ made consisting of eigenvectors for $A$. (Even when all the eigenvalues are real.) When it does happen, we say that $A$ is diagonalizable. Here's an example of a matrix which is NOT diagonalizable:

Exercise 3: Find matrix eigenvalues and eigenspace basis for each eigenvalue, for

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}.$$ 

Explain why there is no basis of $\mathbb{R}^2$ consisting of eigenvectors of $A$. 

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**Output:**

The diagram shows two bases for $\mathbb{R}^3$, with vectors $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (-1, 0, 2)$, and $\mathbf{v}_3 = (1, 1, 0)$. The eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 2$, and $\lambda_3 = 2$. The WolframAlpha input shows the eigenvalues and eigenvectors for the matrix $\begin{bmatrix} 4 & -2 & 1 \\ -2 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$. The output includes the eigenvalues $\lambda_1 = 4$, $\lambda_2 = -2$, and $\lambda_3 = 3$, with corresponding eigenvectors.