2 linear equations in 2 unknowns:

\[ a_{11}x + a_{12}y = b_1 \]
\[ a_{21}x + a_{22}y = b_2 \]

goal: find all \([x, y]\) making both of these equations true at once. Since the solution set to each single equation is a line one geometric interpretation is that you are looking for the intersection of two lines.

**Exercise 3**: Consider the system of two equations \(E_1, E_2\):

\[ E_1 \quad 5x + 3y = 1 \]
\[ E_2 \quad x - 2y = 8 \]

3a) Sketch the solution set in \(\mathbb{R}^2\), as the point of intersection between two lines.

![Graph of L1 and L2](image)

3b) Use the following three "elementary equation operations" to systematically reduce the system \(E_1, E_2\) to an equivalent system (i.e. one that has the same solution set), but of the form

\[ x + 0y = c_1 \]
\[ 0x + 1y = c_2 \]

(so that the solution is \(x = c_1, y = c_2\)). **Make sketches of the intersecting lines, at each stage.**

The three types of elementary equation operation are below. Can you explain why the solution set to the modified system is the same as the solution set before you make the modification?

- interchange the order of the equations
- multiply one of the equations by a non-zero constant
- replace an equation with its sum with a multiple of a different equation.
3c) Look at your work in 3b. Notice that you could have saved a lot of writing by doing this computation "synthetically", i.e. by just keeping track of the coefficients and right-side values. Using $R_1, R_2$ as symbols for the rows, your work might look like the computation below. Notice that when you operate synthetically the "elementary equation operations" correspond to "elementary row operations":

- interchange two rows
- multiply a row by a non-zero number
- replace a row by its sum with a multiple of another row.

\[
\begin{align*}
&x - 2y = 8 \quad \text{lines are the same, we do this step to make it easy to eliminate the}
\vspace{-1em}
\begin{align*}
E_2 & \rightarrow E_2 \\
\frac{13}{-2} \quad y = -3
\end{align*}
\vspace{-1em}
\begin{align*}
2E_2 + E_1 & \rightarrow E_1 \quad x = 2 \\
& y = -3
\end{align*}
\begin{align*}
\text{Solution:} \\
\{x, y\} = \{2, -3\} \\
\text{is the solution to original system:}
\end{align*}
\begin{align*}
x - 2y &= 8 \quad 2 + 6 = 8 \checkmark \\
5x + 3y &= 1 \quad 10 - 9 = 1 \checkmark
\end{align*}
\]
more complete notation:

\[ \begin{array}{ccc}
R_2 & \rightarrow & R_1 \\
R_1 & \rightarrow & R_2 \\
-5R_1 + R_2 & \rightarrow & R_2 \\
\frac{R_2}{13} & \rightarrow & R_2 \\
2R_2 + R_1 & \rightarrow & R_1
\end{array} \]

\[
\begin{array}{cccc}
5 & 3 & | & 1 \\
1 & -2 & \rightarrow & 8 \\
R_1 & 5 & 3 & 1 \\
1 & -2 & \rightarrow & 8 \\
\end{array}
\]

\[
\begin{array}{cccc}
-5R_1 + R_2 & 0 & 13 & -39 \\
1 & -2 & \rightarrow & 8 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 1 & -3 \\
0 & 1 & -3 \\
\end{array}
\]

\[ x = 2 \quad y = -3 \]
3d) What are the possible geometric solution sets to 1, 2, 3, 4 or any number of linear equations in two unknowns?

- empty set (distinct parallel lines)
- single point (all lines have a common point of intersection)
- line (all lines are actually the same line)
- all of \( \mathbb{R}^2 \) (all eqns are \( 0x + 0y = 0 \))
Math 2270-004
Tues Jan 9
• 1.1 Systems of linear equations
• 1.2 Row reduction and echelon form

Announcements:

• finish Monday’s notes
• quiz tomorrow, §1.1, 1.2

Warm-up Exercise:

was something like:

a) Do the lines
\[ E_1 \quad x - 3y = 6 \]
\[ E_2 \quad -2x + 6y = -4 \]
intersect?

b) Explain.

these lines were parallel
( the each had slope \( \frac{1}{3} \) )

\[
\begin{bmatrix}
1 & -3 & 6 \\
-2 & 6 & -4 \\
1 & -3 & 6 \\
\end{bmatrix}
\]

\[ 2R_1 + R_2 \rightarrow R_2 \]
\[
\begin{bmatrix}
1 & -3 & 6 \\
0 & 0 & 8 \\
1 & -3 & 6 \\
\end{bmatrix}
\]

\( 0x + 0y = 8 \) ← says
no solutions!
so the original system has no
solutions “inconsistent”
lines don’t intersect!
Continuing our discussion of solution sets to systems of equations from yesterday:

**Solutions to linear equations in 3 unknowns:**

What is the geometric question we're answering in these cases?

**Exercise 1** Consider the system

\[
\begin{align*}
x + 2y + z &= 4 \\
3x + 8y + 7z &= 20 \\
2x + 7y + 9z &= 23.
\end{align*}
\]

Use elementary equation operations (or if you prefer, elementary row operations in the synthetic version) to find the solution set to this system. There's a systematic way to do this, which we'll talk about today. It's called **Gaussian elimination**. (Check Wikipedia about Gauss - he was amazing!)

**Hint:** The solution set is a single point, \([x, y, z] = [5, -2, 3]\).