Wed Jan 31

- 2.1 Matrix operations

Announcements: quiz ☺

§2.1 Tuesday's notes: matrix multiplication
today's: more matrix algebra

Warm-up Exercise: Let $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be projection from $\mathbb{R}^3$ to the $x_1 - x_2$ plane

a) Find the matrix A so that
$T(\vec{x}) = A\vec{x}$.

b) is T one to one? : NO (not true that $T(\vec{x}) = \vec{b}$ has unique solt'ns (when it has solt'ns)) (there were columns without pivots)

c) is T onto $\mathbb{R}^2$? : YES every row rref(A) has pivot
i.e. $T(\vec{x}) = \vec{b}$ can always solved for $\vec{x}$

a) $$\begin{bmatrix} 1 \cdot x_1 + 0x_2 + 0x_3 \\ 0x_1 + x_2 + 0x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

OR

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$$

standard basis of $\mathbb{R}^3$
$$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

T linear means

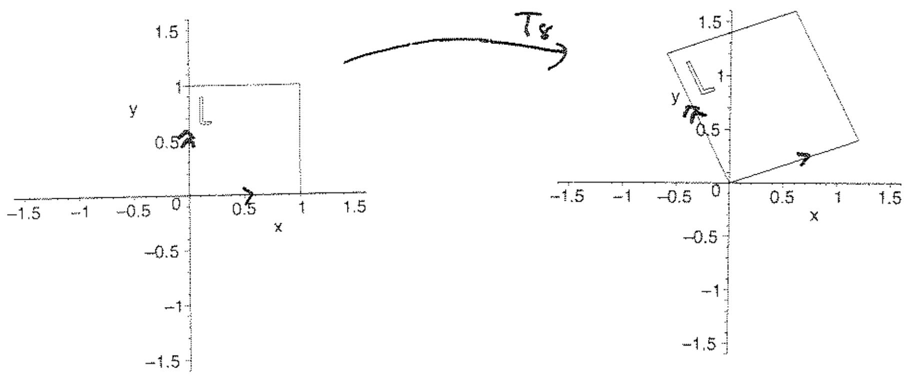
$$T(x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3)$$
$$= T(x_1\vec{e}_1) + T(x_2\vec{e}_2) + T(x_3\vec{e}_3)$$
$$= x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + x_3T(\vec{e}_3)$$
$$= \left[\, T(\vec{e}_1) \,\vdots\, T(\vec{e}_2) \,\vdots\, T(\vec{e}_3) \,\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{(same)}$$

$$\boxed{\begin{array}{l} T \text{ linear} \\ T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \\ T(c\vec{u}) = cT(\vec{u}) \end{array}}$$

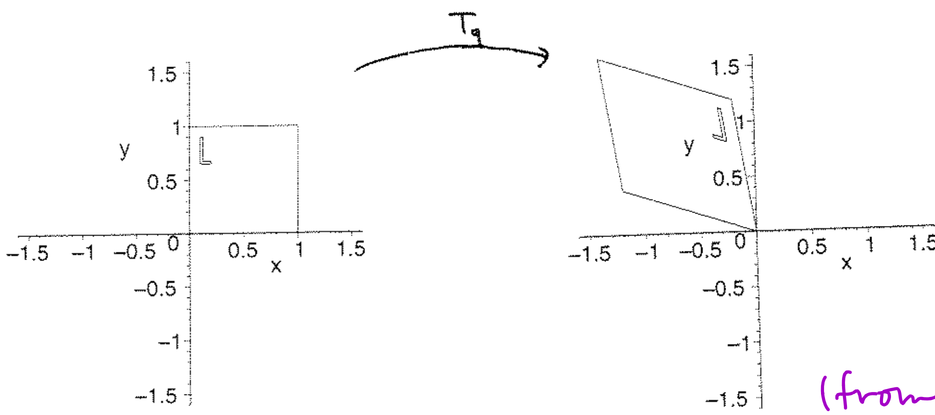
$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(we scrolled back to Monday's notes and found the matrix for T_{10})

mystery linear trans.



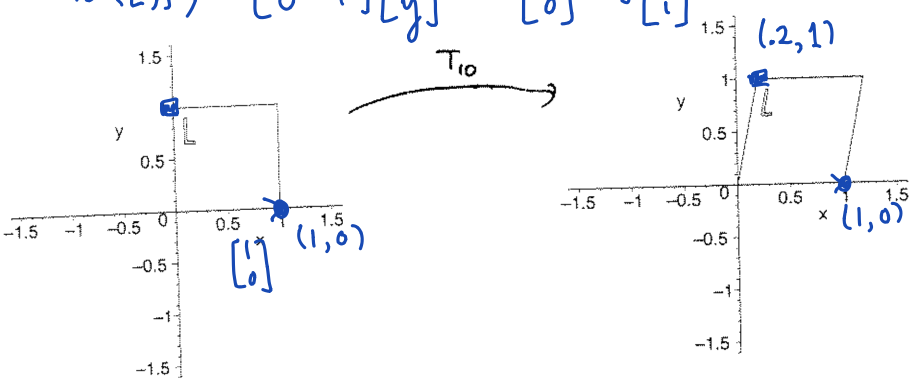
another mystery!



(from "Monday" notes)

so $T_{10}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & .2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} .2 \\ 1 \end{bmatrix}$ Try on Wed.

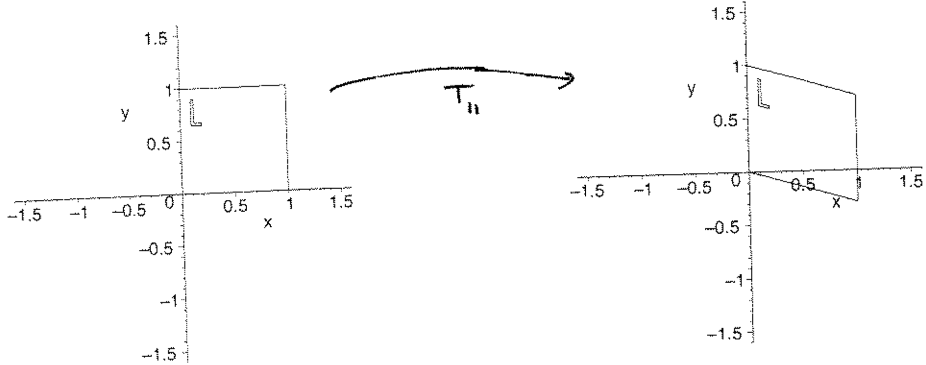
horizontal shear with strength .2



$A = [T(\vec{e}_1) \mid T(\vec{e}_2)]$

$= \begin{bmatrix} 1 & .2 \\ 0 & 1 \end{bmatrix}$

vertical shear with strength −.3



(Then we scrolled to Tuesday's notes and generalized the reasoning from $T: \mathbb{R}^2 \to \mathbb{R}^2$ to $T: \mathbb{R}^n \to \mathbb{R}^m$ linear)

1.9 <u>Theorem</u>: Every linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is actually a matrix transformation!

Some general defintions first, which we've already alluded to:

- In $\mathbb{R}^n$ we write

$$\underline{\boldsymbol{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ : \\ 0 \end{bmatrix}, \ \underline{\boldsymbol{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ : \\ 0 \end{bmatrix}, \ \dots \ \underline{\boldsymbol{e}}_n = \begin{bmatrix} 0 \\ 0 \\ : \\ 1 \end{bmatrix}.$$

- We call

$$\{\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2, \dots \underline{\boldsymbol{e}}_n\}$$

the *standard basis* for $\mathbb{R}^n$ because each

$$\underline{\boldsymbol{x}} = \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

is easily and uniquely expressed as a linear combination of the standard basis vectors,

$$\underline{\boldsymbol{x}} = x_1 \underline{\boldsymbol{e}}_1 + x_2 \underline{\boldsymbol{e}}_2 + \ \dots. \ + x_n \underline{\boldsymbol{e}}_n$$

Example 1:

$$\begin{bmatrix} 3 \\ 7 \\ -6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 6 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 3\underline{\boldsymbol{e}}_1 + 7\underline{\boldsymbol{e}}_2 - 6\underline{\boldsymbol{e}}_3 .$$

you used to call these basis vectors $\hat{i}, \hat{j}, \hat{k}$.

Example 2: For A written in terms of its columns, $A = [\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \dots \underline{\boldsymbol{a}}_n]$,

$$A\,\underline{\boldsymbol{e}}_j = \underline{\boldsymbol{a}}_j \ ,$$

the j^{th} column of A.

$$\left[a_1 \,\vdots\, a_2 \,\vdots\, \cdots \ \vec{a}_n \right] \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{a}_2 .$$

Theorem: Every linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is actually a matrix transformation,

$$T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}},$$

where the j^{th} column of $A_{m \times n}$, is $T(\underline{\boldsymbol{e}}_j), j = 1, 2., \ldots n$. In other words the matrix of T is

$$A = \left[T(\underline{\boldsymbol{e}}_1) \;\; T(\underline{\boldsymbol{e}}_1) \;\; \ldots . \;\; T(\underline{\boldsymbol{e}}_1) \right]$$

proof: T is linear, which means it satisfies

$$(i) \qquad T(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = T(\underline{\boldsymbol{u}}) + T(\underline{\boldsymbol{v}}) \qquad \forall\, \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in \mathbb{R}^n$$

$$(ii) \qquad T(c\,\underline{\boldsymbol{u}}) = c\,T(\underline{\boldsymbol{u}}) \qquad \forall\, c \in \mathbb{R}, \underline{\boldsymbol{u}} \in \mathbb{R}^n .$$

$$T((\vec{u} + \vec{v}) + \vec{w}) \overset{(i)}{=} T(\vec{u} + \vec{v}) + T(\vec{w})$$
$$\overset{(i)}{=} (T(\vec{u}) + T(\vec{v})) + T(\vec{w})$$
$$= T(\vec{u}) + T(\vec{v}) + T(\vec{w})$$

Let's compute $T(\underline{\boldsymbol{x}})$ for $\underline{\boldsymbol{x}} \in \mathbb{R}^n$:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right) = T(x_1\,\underline{\boldsymbol{e}}_1 + x_2\,\underline{\boldsymbol{e}}_2 + \; \ldots . \; + x_n\,\underline{\boldsymbol{e}}_n)$$

$$= T(x_1\,\underline{\boldsymbol{e}}_1) + T(x_2\,\underline{\boldsymbol{e}}_2) + \ldots\, T(x_n\,\underline{\boldsymbol{e}}_n)$$

by repeated applications of the sum property (i).

$$= x_1\,T(\underline{\boldsymbol{e}}_1) + x_2\,T(\underline{\boldsymbol{e}}_2) + \ldots + x_n\,T(\underline{\boldsymbol{e}}_n)$$

by applications fo the scalar multiple property (ii).

$$= A\,\underline{\boldsymbol{x}}$$

for the matrix A given in column form as

$$A = \left[T(\underline{\boldsymbol{e}}_1) \;\; T(\underline{\boldsymbol{e}}_2) \;\; \ldots . \;\; T(\underline{\boldsymbol{e}}_n) \right].$$

Q.E.D.

Exercise 1 Illustrate the linear transformation theorem with the projection function $T : \mathbb{R}^3 \to \mathbb{R}^2$,

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

we did this. warmup Wed.

2.1 Matrix multiplication

Suppose we take a composition of linear transformations:

$$T_1 : \mathbb{R}^n \to \mathbb{R}^m, \quad T_1(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}, \qquad (A_{m\times n}).$$

$$T_2 : \mathbb{R}^m \to \mathbb{R}^p, \quad T_2(\underline{\boldsymbol{y}}) = B\,\underline{\boldsymbol{y}}. \qquad (B_{p\times m})$$

- Then the composition $T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^p$ is linear:

(i)
$$\begin{aligned}(T_2 \circ T_1)(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) &:= (T_2(T_1(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}})))\\ &= T_2(T_1(\underline{\boldsymbol{u}}) + T_1(\underline{\boldsymbol{v}})) && T_1 \text{ linear}\\ &= T_2(T_1(\underline{\boldsymbol{u}})) + T_2(T_1(\underline{\boldsymbol{v}})) && T_2 \text{ linear}\\ &:= (T_2 \circ T_1)(\underline{\boldsymbol{u}}) + (T_2 \circ T_1)(\underline{\boldsymbol{v}})\end{aligned}$$

(ii)
$$\begin{aligned}(T_2 \circ T_1)(c\,\underline{\boldsymbol{u}}) &:= (T_2(T_1(c\,\underline{\boldsymbol{u}})))\\ &= T_2(c\,T_1(\underline{\boldsymbol{u}})) && T_1 \text{ linear}\\ &= c\,T_2(T_1(\underline{\boldsymbol{u}})) && T_2 \text{ linear}\\ &:= c\,(T_2 \circ T_1)(\underline{\boldsymbol{u}})\end{aligned}$$

- So $T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^p$ is a matrix transformation, by the theorem on the previous page.

- Its j^{th} column is

$$\begin{aligned}T_2 \circ T_1(\vec{e}_j) = T_2(T_1(\underline{\boldsymbol{e}}_j)) &= T_2(A(\underline{\boldsymbol{e}}_j))\\ &= T_2(col_j(A))\\ &= B(col_j(A)).\end{aligned}$$

- i.e. the matrix of $T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^p$ is

$$[B\,\underline{\boldsymbol{a}}_1 \; B\,\underline{\boldsymbol{a}}_2 \; \ldots \; B\,\underline{\boldsymbol{a}}_n] := B\,A.$$

where $col_j(A) = \underline{\boldsymbol{a}}_j$.

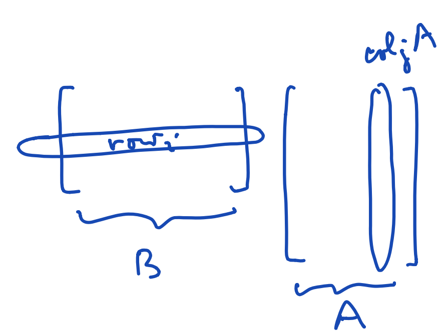
<u>Summary</u>: For $B_{p\times m}$ and $A_{m\times n}$

- the matrix product $(BA)_{p\times n}$ is defined by

$$col_j(B\,A) = B\,col_j(A) \qquad j = 1..n$$

- or equivalently

$$entry_{ij}(B\,A) = row_i(B) \cdot col_j(A) \quad i = 1..p, j = 1..n$$



- And,

$$(BA)\underline{\boldsymbol{x}} = B\,(A\,\underline{\boldsymbol{x}})$$

because BA is the matrix of $T_2 \circ T_1$.

$entry_{ij}(BA)$
$= row_i(B) \cdot col_j(A)$

Exercise 2 Compute

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & -2 & 3 \\ 1 & 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 6 & 5 \\ 3 & -1 & 14 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

or $\begin{bmatrix} 0\cdot 1 + 1\cdot 2 \\ -1\cdot 0 + 3\cdot 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\rightarrow$ 3rd col. $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2+8 \\ 2+12 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$

Exercise 3 For

$$T_1(\underline{\boldsymbol{x}}) = \begin{bmatrix} 0 & 1 & -2 & 3 \\ 1 & 0 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad T_2(\underline{\boldsymbol{y}}) = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

compute $T_2(T_1(\underline{\boldsymbol{x}}))$. How does this computation relate to Exercise 2?